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ON NADLER'S MULTI-VALUED CONTRACTION PRINCIPLE IN COMPLETE METRIC SPACES*

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Dedicated to Professor Mihail Megan
on the occasion of his 70th anniversary

Abstract

The aim of this paper to present an extended variant of the multi-valued contraction principle. Under the classical assumptions considered by Nadler (1969) and Covitz and Nadler (1970) (i.e., the completeness of the metric space (X, d) and the contraction assumption on a self multi-valued operator on X having nonempty and closed values) several other conclusions with respect to the fixed point problem are presented.

MSC: 47H10, 54H25.

keywords: multi-valued operator, complete metric space, fixed point, strict fixed point, data dependence, Ulam-Hyers stability, well-posedness, Ostrovski property, qualitative properties of the fixed point set.

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1 Introduction

It is known that the main metric fixed point theorems for multi-valued contractions on a complete metric space were given by Nadler [11] and Covitz-Nadler [4] (see also [3]). Several extensions of this principle were considered by some authors to different types of generalized metric spaces (see e.g. [5], [13], [6], ...) and to some generalized contractions conditions (see e.g. [26], [22], [24], [27], [28], [30], [31], ...).

Moreover, under the basic assumptions of the multi-valued contraction principle, several other properties of the fixed point set were obtained in the last decades. The aim of this paper to present an extended (by the conclusions point of view) version of the multi-valued contraction principle. For the single-valued case see [25]. For more details on this subject, see [14].

2 Preliminaries

Let us recall first some important preliminary concepts and results.

Let (X, d) be a metric space and $P(X)$ be the family of all nonempty subsets of X. We denote by $P_{cl}(X)$ the family of all nonempty closed subsets of X, by $P_b(X)$ the family of all nonempty bounded subsets of X and by $P_{cp}(X)$ the family of all nonempty compact subsets of X. For $x_0 \in X$ and $r > 0$ we will also denote by $B(x_0; r) := \{x \in X | d(x_0, x) < r\}$ the open ball centered in x_0 with radius r.

We also recall, in the context of a metric space, the definitions of some important functionals in multi-valued analysis theory:
(a) the gap functional generated by d:

$$D_d : P(X) \times P(X) \to \mathbb{R}_+, \ D_d(A, B) := \inf\{d(a, b) \mid a \in A, \ b \in B\};$$

(b) the excess functional of A over B generated by d:

$$e_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \ e_d(A, B) := \sup\{D_d(a, B) \mid a \in A\};$$

(c) the Hausdorff-Pompeiu functional generated by d:

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \ H_d(A, B) = \max\{e_d(A, B), e_d(B, A)\}.$$

The diameter of a set $Y \in P(X)$ will be denoted by $diam(Y) := \sup\limits_{x,y \in Y} d(x, y)$.

Some useful properties of these functionals are re-called (see, for example, [2], [7], [13], [17],) in the next lemma. These properties are important tools in the proof of the main conclusions of the Nadler's multi-valued contraction principle.

Lemma 1 *If (X,d) is a metric space, then we have:*

(a) H_d is a generalized (in the sense that H could also take the value $+\infty$, see [9]) metric in $P_{cl}(X)$;

(b) if $A, B \in P(X)$ and $q > 1$, then, for every $a \in A$ there exists $b \in B$ such that $d(a,b) \leq qH_d(A,B)$.

(c) if there exists $\eta > 0$ such that for each $a \in A$ there exists $b \in B$ such that $d(a,b) \leq \eta$, then $e_d(A,B) \leq \eta$.

(d) if $A \in P(X)$ and $b \in X$ are such that $D_d(b,A) = 0$, then $b \in \overline{A}$.

Recall that if (X,d) is a metric space, then a set $Y \in P(X)$ is said to be proximinal if for every $x \in X$ there exists $y \in Y$ such that $d(x,y) = D(x,Y)$.

Finally, let us recall that if X is a nonempty set and $F : X \to P(X)$ is a multi-valued operator, then we denote by $Fix(F) := \{x \in X : x \in F(x)\}$ the fixed point set for F, by $SFix(F) := \{x \in X : \{x\} = F(x)\}$ the strict fixed point set for F, by $Graph(F) := \{(x,y) \in X \times X | y \in F(x)\}$ the graph of F and by $I(F) := \{Y \subset X : F(Y) \subset Y\}$ the set of all invariant subsets of X with respect to F.

Moreover, for arbitrary $(x_0, x_1) \in Graph(F)$, the sequence $(x_n)_{n\in\mathbb{N}}$ with $x_{n+1} \in F(x_n)$ (for $n \in \mathbb{N}^*$) is called the sequence of successive approximations for F staring from (x_0, x_1).

Let (X,d) be a metric space and $F : X \to P(X)$ be a multi-valued operator. Then, if $Y \in P(X)$ we denote by $F(Y) := \bigcup_{y\in Y} F(y)$ the image of the set Y through F. We also denote by

$$F^0 := 1_X,\ F^1 := F, \ldots, F^{n+1} = F \circ F^n,\ n \in \mathbb{N}$$

the iterate operators of F, where $(F \circ F)(Y) := F(F(Y))$, for $Y \in P(X)$. In the same framework, the set-to-set operator $\hat{F} : P(X) \to P(X)$, defined by

$$\hat{F}(Y) := \bigcup_{x\in Y} F(x),\ \text{for } Y \in P(X)$$

is called Nadler's set-to-set operator induced by F.

Some typical conditions in fixed point theory for a multi-valued operator are given now.

Definition 1 *Let (X,d), (Y,d') be metric spaces and $F : X \to P(Y)$. Then, F is called an α-contraction if $\alpha \in (0,1)$ and $H(F(x_1),F(x_2)) \leq \alpha d(x_1,x_2)$, for all $x_1,x_2 \in X$.*

Lemma 2 *Let (X,d) be a metric space and $F : X \to P_{cl}(X)$ be α-contraction. Then, $Graph(F)$ is a closed set in the topology of $X \times X$.*

The concept of multi-valued weakly Picard operator is central in our approach.

Definition 2 *([27, 28, 16]) Let (X,d) be a metric space. Then $F : X \to P(X)$ is called a multivalued weakly Picard operator (briefly, MWP operator) if for each $x \in X$ and each $y \in F(x)$ there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that*

(i) $x_0 = x,\ x_1 = y$;

(ii) $x_{n+1} \in F(x_n)$, for all $n \in \mathbb{N}$;

(iii) the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

Let us recall the following important notion.

Definition 3 *Let (X,d) be a metric space and $F : X \to P(X)$ be an MWP operator. Then we define the multivalued operator $F^\infty : Graph(F) \to P(Fix(F))$ by the formula $F^\infty(x,y) = \{z \in Fix(F) \mid$ there exists a sequence of successive approximations of F starting from (x,y) that converges to $z\}$.*

An important concept is given by the following definition.

Definition 4 *Let (X,d) be a metric space and $F : X \to P(X)$ an MWP operator. Then F is a ψ-multi-valued weakly Picard operator (briefly ψ-MWP operator) if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 with $\psi(0) = 0$ and there exists a selection f^∞ of F^∞ such that*

$$d(x, f^\infty(x,y)) \leq \psi(d(x,y)), \text{ for all } (x,y) \in Graph(F).$$

We will also recall now the notion of multi-valued Picard operator.

Definition 5 *([16], [17]) We say that $F : X \to P(X)$ is a multi-valued Picard operator if:*

(i) $SFix(F) = Fix(F) = \{x^*\}$;

(ii) $F^n(x) \overset{H_d}{\to} \{x^*\}$ as $n \to \infty$, for each $x \in X$.

Several examples of Picard and weakly Picard operators, as well as, different applications of this theory are given, for example, in [13], [16], [15], [17], [19].

3 Multi-valued contraction principle: two extended versions

In 1969, S.B. Nadler Jr. proved the first metric fixed point principle for multi-valued operators in complete metric spaces. Then, in 1970 S. Covitz and S.B. Nadler Jr. proved a slight generalizations of it. This theorem is usually known in the literature as Multivalued Contraction Principle (MCP).

Theorem 1 (**MCP***(1969, 1970))* *Let* (X, d) *be a complete metric space and let* $x_0 \in X$*. If* $F : X \to P_{cl}(X)$ *is a multivalued* α*-contraction, then* $Fix(F) \neq \emptyset$ *and there exists a sequence of successive approximations for* F*, starting from* x_0*, which converges to a fixed point of* F*.*

Later on, it was noticed in several papers that there are several other conclusions which follow by the main assumptions (completeness of the metric space and the contraction condition for a self multi-valued operator with nonempty and closed values) of MCP. More precisely, we have the following extended version of the fixed point principle for multi-valued contractions.

Theorem 2 (An extended version of the MCP) *Let* (X, d) *be a complete metric space and* $F : X \to P_{cl}(X)$ *be a multi-valued* α*-contraction. Then, the following conclusions hold:*

(a) there exists $x^* \in X$ *such that* $x^* \in Fix(F^n)$ *for each* $n \in \mathbb{N}^*$*;*

(b) there exists a sequence of successive approximations for F*, starting from any pair* $(x, y) \in Graph(F)$*, which converges to a fixed point* $f^\infty(x, y)$ *of* F*;*

(c) $Fix(F)$ *is closed in* (X, d)*;*

(d) there exists a selection $f^\infty : Graph(F) \to Fix(F)$ *of* F^∞ *such that*

$$d(x, f^\infty(x, y)) \leq \psi(d(x, y)), \text{ for all } (x, y) \in Graph(F),$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) = \frac{1}{1-\alpha}t$*;*

(e) if $(x_n)_{n\in\mathbb{N}}$ *is a sequence of successive approximations for* F*, starting from any pair* $(x, y) \in Graph(F)$*, which converges to a fixed point* $f^\infty(x, y)$ *of* F*, then*

$$d(x_n, f^\infty(x, y)) \leq \frac{\alpha^n}{1-\alpha} d(x, y), \text{ for each } n \in \mathbb{N}^*;$$

(f) if $G : X \to P_{cl}(X)$ *is a multi-valued* β*-contraction and* $\eta > 0$ *is such that*

$$H(F(x), G(x)) \leq \eta, \text{ for each } x \in X,$$

then

$$H(Fix(F), Fix(G)) \leq \frac{\eta}{1 - \max\{\alpha, \beta\}};$$

(g) if $F_n : X \to P_{cl}(X)$, $n \in \mathbb{N}$ *is a sequence of multi-valued* α*-contractions such that* $F_n(x) \overset{H}{\to} F(x)$ *as* $n \to +\infty$*, uniformly with respect to* $x \in X$*, then*

$$Fix(F_n) \overset{H}{\to} Fix(F) \text{ as } n \to +\infty, \text{ i.e., } \lim_{n\to\infty} H(Fix(F_n), Fix(F)) = 0;$$

(h) if there exists $x_0 \in X$ *and* $r > 0$ *such that* $D(x_0, F(x_0)) < (1-\alpha)r$*, then there exists* $x^* \in Fix(F) \cap B(x_0; r)$*;*

(i) in particular, if X *is a Banach space, then the associated multi-valued field* $G(x) := x - F(x)$ *is open and surjective;*

(j) the operator $\hat{F} : P_{cl}(X) \to P_{cl}(X)$ *defined by* $\hat{F}(Y) := \overline{\bigcup_{x\in Y} F(x)}$ *is an* α*-contraction with respect to* H_d*, i.e.,*

$$H_d(\hat{F}(A), \hat{F}(B)) \leq \alpha H_d(A, B), \ \forall\ A, B \in P_{cl}(X) \text{ with } H_d(A, B) < +\infty,$$

and, if additionally there exists $A_0 \in P_{cl}(X)$ *such that* $H_d(A_0, F(A_0)) < +\infty$*, then there exists at least one* $A_F^* \in P_{cl}(X)$ *such that* $\hat{F}(A_F^*) = A_F^*$*;*

(k) in particular, if F *has proximinal values, then for any* $\epsilon > 0$ *and any* ϵ*-solution* z *of the fixed point problem* $x \in F(x)$ *(i.e.,* $D(z, F(z)) \leq \epsilon$*) there exists* $x^* \in Fix(F)$ *such that* $d(z, x^*) \leq \psi(\epsilon)$*, where* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$*,* $\psi(t) = \frac{1}{1-\alpha}t$*;*

(l) in particular, if F *has compact values, then* $\hat{F} : P_{cp}(X) \to P_{cp}(X)$ *and it is an* α*-contraction. Additionally, the following conclusions hold:*

(l_1) $Fix(\hat{F}) = \{A_F^*\}$*;*

(l_2) $F^n(Y) \overset{H}{\to} A_F^*$ *as* $n \to +\infty$*, for each* $Y \in P_{cp}(X)$*;*

(l_3) $A_F^* = \overline{\bigcup_{n\in\mathbb{N}^*} F^n(x)}$*, for each* $x \in Fix(F)$*;*

(l_4) $Fix(F) \subset A_F^*$*;*

(l_5) $Fix(F)$ *is compact in* (X, d)*.*

(m) in particular, if X *is a closed convex subset of a Banach space and, additionally,* F *has convex values, then* $Fix(F)$ *is arcwise connected;*

(n) if, additionally, (X, d) *is a convex metric space and* F *has bounded values, then, for any* $x^* \in Fix(F)$*, we have* $diam(Fix(F)) \leq \frac{1}{1-\alpha} \cdot diam(F(x^*))$*;*

(p) there exists a Caristi selection of F;

(r) if, for $p > 0$, we denote $Fix_p(F) := \{x \in X : D(x, F(x)) < p\}$, then the following relation holds

$$H(Fix_p(F), Fix(F)) \leq \frac{p}{1-\alpha}.$$

Let us present now some remarks about some of the above assertions.

1) Most of the above assertions are known in fixed point theory. We refer to [6], [8], [10], [12], [16], [17], [18], [23] for other considerations.

2) By definition, a multi-valued operator satisfying (a) and (b) is called a multi-valued wealy Picard operator, see Definition 2. If additionally, if F satisfies (a), (b) and (d) then the operator F is called a ψ-MWP, see Definition 4.

3) The conclusion (f) shows that the data dependence phenomenon for the fixed point set of a multi-valued contraction takes place.

4) Conclusion (k) is known as the Ulam-Hyers stability property of the fixed point problem $x \in F(x)$.

A special case of the above principle is the following extended version of the strict fixed point principle for multi-valued contractions. We note that most of the above conclusions are known in fixed point theory.

Theorem 3 (An extended version of the MCP with respect to strict fixed points)

Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a multi-valued α-contraction such that $SFix(F) \neq \emptyset$. Then, the following conclusions hold:

(a) $Fix(F) = SFix(F) = \{x^*\}$;

(b) $Fix(F^n) = SFix(F^n) = \{x^*\}$ for $n \in \mathbb{N}$ with $n \geq 2$;

(c) $F^n(x) \overset{H}{\to} \{x^*\}$ as $n \to +\infty$, for each $x \in X$;

(d) if $G : X \to P_{cl}(X)$ is a multi-valued operator with $Fix(G) \neq \emptyset$ and there exists $\eta > 0$ such that $H(F(x), G(x)) \leq \eta$ for each $x \in X$, then $H(Fix(F), Fix(G)) \leq \frac{\eta}{1-\alpha}$;

(e) Let $F_n : X \to P_{cl}(X)$, $n \in \mathbb{N}$ be a sequence of multivalued operators such that $Fix(F_n) \neq \emptyset$ for each $n \in \mathbb{N}$ and $F_n(x) \overset{H}{\to} F(x)$ as $n \to +\infty$, uniformly with respect to $x \in X$. Then $Fix(F_n) \overset{H}{\to} \{x^*\}$ as $n \to +\infty$;

(f) if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $D(x_n, F(x_n)) \to 0$ as $n \to \infty$, then $x_n \overset{d}{\to} x^*$ as $n \to \infty$;

(g) if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $H(x_n, F(x_n)) \to 0$ as $n \to \infty$, then $x_n \overset{d}{\to} x^*$ as $n \to \infty$;

(h) if $(y_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $D(y_{n+1}, F(y_n)) \to 0$ *as* $n \to \infty$*, then* $y_n \overset{d}{\to} x^*$ *as* $n \to \infty$*.*

Let us present now some remarks about some of the above assertions.

1) By definition, a multi-valued operator satisfying (a) and (c) is called a multi-valued Picard operator, see Definition 5.

2) Conclusion (d) is a data dependence theorem for the strict fixed point with respect to an arbitrary perturbation of the multi-valued operator F.

3) The conclusions (f)-(g) give the well-posedness property of the fixed point problem with respect to D_d and, respectively, with respect to H_d.

4) Conclusion (h) is known as the Ostrovski property of the fixed point problem $x \in F(x)$.

Conclusions. The above theorems contain the most important consequences of the MCP. These are important since it is well-known that many operator type inclusions in applied mathematics (integral and differential inclusions, equilibrium problems, game theory problems, mathematics of fractals, see [1], [20], [21] for some recent results for the single-valued and the multi-valued case too) are reduced to a fixed point (or a strict fixed point) inclusion for an appropriate multi-valued operator. Thus, the above conclusions induce similar properties for the solutions of operator inclusion problem. For more details, extensions and generalizations of the above results see [14].

Open Problems. 1) It is an open question to give explicit conditions for the existence of at least one strict fixed point of a multi-valued contraction with closed values. This problem is important not only from the above theorem point of view, but also because many iteration methods for a multi-valued operator F are working under the assumption that $SFix(F) \neq \emptyset$.
2) Another open problem is to extend the above results to different classes of multi-valued generalized contractions.

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