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APPLICATIONS OF HAMILTONIAN SYSTEMS IN ANALYSIS AND OPTIMIZATION*

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Dedicated to Professor Mihail Megan
on the occasion of his 70th anniversary

Abstract

In this paper we present a short description of the PhD thesis with the title "Applications of Hamiltonian systems in analysis and optimization". This thesis was defended on November 2-nd 2017 at the Institute of Mathematics "Simion Stoilow" of the Romanian Academy.

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1 Preliminary notions about implicit systems

Implicit functions represent a classical subject, still very studied. One of the first references that includes the modern formulation of the theorem is Dini, 1878, [9].

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A new approach (constructive), the implicit parametrization method, was introduced in the paper Tiba, 2013, [33], in dimension two and three. An advantage of this method is that it uses just systems of ordinary differential equations to obtain the solution in parametric form. It can be applied in the critical case too.

In this chapter, we have presented important results form the recent literature, referring to the implicit function theorem. We discuss the constructive approach and we present the constructive reformulation of the classical implicit function theorem Diener, Schuster, 2009, [8], and also the case of power series Torriani, 1989, [35], Sokal, 2009, [31].

We give results on the local character, about the size of the neighborhood where the solution exists, Holtzman, 1970, [12], Chang, He, Prahbu, 2003, [5], Phien, 2012, [29], and different global formulations for the implicit function theorem, Zhang, Ge, 2006, [36], Cristea, 2007, [7], Idczak, [14], [15], [16]. In the last part of this chapter we discuss the case when the differentiability hypothesis from the implicit function theorem is replaced by weaker ones, Jittorntrum, 1978, [17], Kumagai, 1980, [19], Clarke, 1983, [6], Hurwicz, Richter, 2006, [13], Dontchev, Rockafellar, 2009, [10].

2 Hamiltonian Method

2.1 Curves in dimension two and three

Here, we follow first Tiba, 2013, [33].

Consider the implicit equation (1), where $\Omega \subset \mathbb{R}^2$ is an open subset and $g : \Omega \to \mathbb{R}$ is of class $C^1(\Omega)$:

$$g(x, y) = 0 \text{ in } \Omega, \tag{1}$$

and we assume the classical conditions

$$\begin{aligned} g(x_0, y_0) &= 0 \\ \nabla g(x_0, y_0) &\neq 0. \end{aligned}$$

Here $\nabla g(x, y)$ is the normal vector to the level lines of g. Therefore, vector (2) gives the tangent to the curve (defined by the implicit function theorem):

$$tg(x, y) = \left(-\frac{\partial g}{\partial y}(x, y), \frac{\partial g}{\partial x}(x, y) \right) \neq 0. \tag{2}$$

We define the Hamiltonian system (3) with the initial conditions (4):

$$\begin{aligned} x'(t) &= -\frac{\partial g}{\partial y}(x(t), y(t)) \\ y'(t) &= \frac{\partial g}{\partial x}(x(t), y(t)) \end{aligned} \tag{3}$$

$$x(0) = x_0, y(0) = y_0. \tag{4}$$

Proposition 1. *We have:*

$$g(x(t), y(t)) = 0, \ \forall \ t \in I_{max},$$

where I_{max} is the maximal existence interval for the problem (3), (4), according to Peano's theorem.

We consider now the implicit system in dimension three, Tiba, 2013, [33]:

$$F(x, y, z) = 0, \ G(x, y, z) = 0, \tag{5}$$

where $F, G : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ are of class $C^1(\Omega)$, Ω is an open subset of $\mathbb{R}^3$ and:

$$\frac{D(F, G)}{D(y, z)} \neq 0, \quad \text{in} \ \ (x_0, y_0, z_0).$$

We assume that (5) and the independence condition are satisfied in $(x_0, y_0, z_0) \in \Omega$.

We will build an explicit parametrization which solves (5), using the normal vectors to the two surfaces around (x_0, y_0, z_0): $n_1 = \nabla F(x, y, z)$, $n_2 = \nabla G(x, y, z)$ and the tangent vector to the curve obtained from their intersection: $\theta = n_1 \times n_2$.

We now consider the ordinary differential system:

$$\begin{aligned} &x'(t) = \theta_1(t) \\ &y'(t) = \theta_2(t) \\ &z'(t) = \theta_3(t) \\ &x(0) = x_0, y(0) = y_0, z(0) = z_0. \end{aligned} \tag{6}$$

From Peano's theorem, the system (6) has at least one solution defined on some maximal existence interval I_{max} around 0.

Proposition 2. *We have:*

$$F(x(t), y(t), z(t)) = G(x(t), y(t), z(t)) = 0, \ \ \forall t \ \in I_{max}.$$

2.2 Surfaces in dimension three

In dimension three, consider the case of just one implicit equation:

$$f(x, y, z) = 0, \tag{7}$$

where $f \in C^1(\overline{\Omega})$, $\Omega \subset \mathbb{R}^3$ is a bounded domain and we have $f(x_0, y_0, z_0) = 0$. We obtain a parametrization for surface (7) using just ordinary differential systems.

We associate with the implicit equation (7) two iterated Hamiltonian systems, Nicolai, Tiba, 2015, [26]:

$$\begin{aligned} x'(t) &= -f_y(x(t), y(t), z(t)), & t \in I_1, \\ y'(t) &= f_x(x(t), y(t), z(t)), & t \in I_1, \qquad (8) \\ z'(t) &= 0, & t \in I_1, \\ x(0) &= x_0, \ y(0) = y_0, \ z(0) = z_0; & \qquad (9) \end{aligned}$$

$$\begin{aligned} \dot{\varphi}(s,t) &= -f_z(\varphi(s,t), \psi(s,t), \xi(s,t)), & s \in I_2(t), \\ \dot{\psi}(s,t) &= 0, & s \in I_2(t), \qquad (10) \\ \dot{\xi}(s,t) &= f_x(\varphi(s,t), \psi(s,t), \xi(s,t)), & s \in I_2(t), \\ \varphi(0,t) &= x(t), \ \psi(0,t) = y(t), \ \xi(0,t) = z(t), & \qquad (11) \end{aligned}$$

where f_x, f_y, f_z are the derivatives of $f(\cdot,\cdot,\cdot)$, with respect to x, y and z and I_1, $I_2(t)$ are real closed intervals, containing 0 in their interior. We impose that the initial condition satisfies: $f(x_0, y_0, z_0) = 0, \quad \nabla f(x_0, y_0, z_0) \neq 0$.

In fact, we choose $f_x(x_0, y_0, z_0) \neq 0$, which ensures that (x_0, y_0, z_0) is not a critical point for both systems.

If we have $f_y(\tilde{x}, \tilde{y}, \tilde{z}) \neq 0$, we can also use a second variant of the iterated Hamiltonian systems:

$$
\begin{aligned}
x'(t) &= -f_y(x(t),y(t),z(t)), & t \in I_1,\\
y'(t) &= f_x(x(t),y(t),z(t)), & t \in I_1,\\
z'(t) &= 0, & t \in I_1,\\
x(0) &= \tilde{x},\ y(0) = \tilde{y},\ z(0) = \tilde{z};
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
\dot{\varphi}(s,t) &= 0, & s \in I_2(t)\\
\dot{\psi}(s,t) &= -f_z(\varphi(s,t),\psi(s,t),\xi(s,t)), & s \in I_2(t),\\
\dot{\xi}(s,t) &= f_y(\varphi(s,t),\psi(s,t),\xi(s,t)), & s \in I_2(t),\\
\varphi(0,t) &= x(t),\ \psi(0,t) = y(t),\ \xi(0,t) = z(t).
\end{aligned}
\tag{13}
$$

We observe that if $\nabla f(\tilde{x},\tilde{y},\tilde{z}) \neq 0$, for a good choice of the axes, we can obtain $f_x(\tilde{x},\tilde{y},\tilde{z}) \neq 0$ and $f_y(\tilde{x},\tilde{y},\tilde{z}) \neq 0$ and we can use both variants of the systems. The obtained solutions are local around the initial condition. In dimension three, the solution of the alternative system (12) - (13) can be taken together with the one of the first Hamiltonian system (8) - (11), to obtain more information.

An important property of the existence intervals of the systems (8) - (11) is that they can be chosen independently of the parameter t.

Proposition 3. *[Nicolai, Tiba,* 2015*, [26]] The following two affirmations are true:*
a) $I_2(t) \supset I_2,\ \ \forall t \in I_1$, *where* $0 \in int\, I_2$ *and* $I_2 = \overline{I}_2 \subset \mathbb{R}$, *i.e. systems (8) - (11) can be chosen independently of the parameter* t.
b) systems (8), (9), *respectively* (10), (11) *have unique solutions on* I_1, *respectively on* I_2, *for any fixed* $t \in I_1$, *in the class of equivalent parametrizations.*

The following two results were first presented in Nicolai, Tiba, 2015, [26].

Corollary 1.

$$f(\varphi(s,t),\psi(s,t),\xi(s,t)) = f(x(t),y(t),z(t)) = 0,\ \ \forall\ \ (s,t) \in I_1 \times I_2. \tag{14}$$

Proposition 4. *If we choose the domain* $I_1 \times I_2$ *small enough around the origin and* $f \in C^2(\Omega)$, *the mapping* $(\varphi,\psi,\xi) : I_1 \times I_2 \to R^3$ *is regular and one-to-one on its image.*

For numerical examples we recommend Nicolai, Tiba, 2015, [26] and Nicolai, 2015, [24].

3 Critical case

3.1 Dependence on the initial data

The varieties that represent the solutions of implicit systems in the classical case, can be obtained as the solution of a Hamiltonian systems in dimension two, or of iterated Hamiltonian systems, in dimension three. The well known continuity properties of the initial conditions of ordinary differential systems are also true for iterated systems, and they play a fundamental role in defining the notion of generalized solution, in the case where the classical hypothesis of independence is fulfilled. This chapter is based on the original papers Nicolai, Tiba, 2015, [26], Nicolai, 2015, [23] and also we use the papers Tiba, 2013, [33], Ombach, 1970, [28], Krantz, Parks, 2002, [18].

3.1.1 Generalized solutions

We now assume that $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ is $C^1(\bar{\Omega})$ and ∇f is locally Lipschitzian in Ω. We will treat the critical case, in dimension two and three.

In dimension three, we consider the case of one implicit equation and a critical initial point $(x_0, y_0, z_0) \in \Omega$:

$$\begin{aligned} f(x_0, y_0, z_0) &= 0, \\ \nabla f(x_0, y_0, z_0) &= 0. \end{aligned}$$

The following results represent a continuity property for systems (8) - (11), that was introduced in Nicolai, Tiba, 2015, [26].

Proposition 5. *Let $(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \to (x_0, y_0, z_0) \in \Omega$ and (x_n, y_n, z_n), $(\varphi_n, \psi_n, \xi_n)$ be the solutions of (8) - (11) associated to the initial condition $(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n)$.*

The next statements are true:

i) There exists two close intervals $\tilde{I}_1$, $\tilde{I}_2 \subset \mathbb{R}$, containing 0 in their interior and (x_n, y_n, z_n) defined on $\tilde{I}_1$, respectively $(\varphi_n, \psi_n, \xi_n)$ defined on $\tilde{I}_1 \times \tilde{I}_2$, for all $n \in N$,

ii) $(x_n, y_n, z_n) \to (x, y, z)$ in $C^1(\tilde{I}_1)^3$,

iii) $(\varphi_n, \psi_n, \xi_n) \to (\varphi, \psi, \xi)$ in $C(\tilde{I}_1 \times \tilde{I}_2)^3$.

To obtain the *generalized solution* for (7) in the critical case, we consider $(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \to (x_0, y_0, z_0)$ in Ω, such that $\nabla f(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n) \neq 0$.

Let $(\varphi_n, \psi_n, \xi_n) : \tilde{I}_1 \times \tilde{I}_2 \to \mathbb{R}$ be the solution of systems (8) - (11) corresponding to the initial condition $(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n)$ and we consider the set:

$$T_n = \{(\varphi_n(s,t), \psi_n(s,t), \xi_n(s,t)) : (s,t) \in \tilde{I}_1 \times \tilde{I}_2\}.$$

$T_n \subset \mathbb{R}^3$ is a compact subset since $\tilde{I}_1$, $\tilde{I}_2$ are compact intervals, $T_n \subset \Omega$ bounded, for all $n \in \mathbb{N}$, since Ω is bounded. On a subsequence, denoted by n, we have: $T_n \to T_\alpha$ in the Hausdorff-Pompeiu metric, Tiba, 2013, [33], Neittaanmaki, Sprekels, Tiba, 2006, [22], where T_α is some compact in $\mathbb{R}^3$ (α being a notation for the subsequence).

We can now introduce the following definition, as in Nicolai, Tiba, 2015, [26].

Definition 1. *We define the local generalized solution of equation (7) as the set:*

$$T = \cup_{\alpha \in \Lambda} T_\alpha,$$

where Λ is the family of all sequences and subsequences satisfying this properties.

Proposition 6. *We have $(x_0, y_0, z_0) \in T$ and for any $(\tilde{x}, \tilde{y}, \tilde{z}) \in T$, it results that $f(\tilde{x}, \tilde{y}, \tilde{z}) = 0$.*

Numerical examples are presented in Nicolai, Tiba, 2016, [27].

3.2 An algorithm for the computation of the generalized solution for implicit systems

We study the approximation of the solution for the implicit function systems, in the critical case. The method we will use is based on iterative systems of ordinary differential equations, to obtain the solution in parametric form.

Similar algorithms may be formulated for general implicit systems, but in this paper we consider just the scalar case and for $d = 2$ or $d = 3$. We present the steps of the algorithm that was introduced in Nicolai, 2015, [23].

Algorithm 3.3.1

Step 1: *We choose $\varepsilon > 0$ and a division of a neighborhood of the initial condition, of dimension ε. For this neighborhood, one can consider a sphere or a cube of "dimension" $\varepsilon > 0$ and we divide it in k parts.*

Step 2: *For each of the divisions we take one point that we will take as the initial condition of the differential system (3) in dimension two and for (8) - (11), in dimension three.*

Step 3: *For the next step we divide the neighborhood in $2k$ parts or we can consider a smaller neighborhood, for example by taking the dimension of the sphere / cube $\varepsilon/2$.*

Step 4: We again compute the approximate solutions for (3) or (8) - (11) for the new chosen neighborhood.

Step 5: After two consecutive iterations, we compute the Hausdorff-Pompeiu distance between the new solutions. We limit the obtained solution to a neighborhood of the initial condition, that we fix from the beginning.

Step 6: We compare the obtained Hausdorff-Pompeiu distance with a certain value, that we also fix from the beginning, and if it is grater that this tolerance, we return to Step 3. If not, we end the algorithm.

For the stopping criterion one can use different conditions. For example, we can also fix from the beginning a maximum number of iterations.

4 Applications in analysis

In Chapter 4 we present applications of the implicit parametrization method in analysis, Nicolai, 2015, [24]. We discuss the Lagrange equation combining our methodology with a method from Mirică, 1989, [21], § 1.6.

4.1 Implicit differential equations

In this section we study the Lagrange differential equation using the implicit parametrization method.

For this we consider the next Lagrange equation (15), Barbu, 1985, [2], Mirică, 1989, [21]:

$$x = ta(x') + b(x'), \tag{15}$$

where $a(\cdot),\ b(\cdot) \in C^1(\mathbb{R})$ and the initial condition (16):

$$x(t_0) = x_0. \tag{16}$$

We use the implicit parametrization method, Nicolai, 2015, [24], introduce the notation $x' = y$ and we look for a parametrization for the solution of the equation:

$$f(t, x, y) = 0, \tag{17}$$

where

$$f : \mathbb{R}^3 \to \mathbb{R},\ f(t, x, y) = x - ta(y) - b(y),\ f(t_0, x_0, y_0) = 0. \tag{18}$$

After laborious calculus, that can be consulted in Nicolai, 2015, [24], and using a result from Mirica, 1989, [21], we get a linear differential equation, which we can solve. We obtain the following solution (in parametric form):

$$
\begin{aligned}
t(\theta) &= exp\left(\int_0^\theta a'\left(A^{-1}(\tau + A(y_0))\right)d\tau\right)\Big((t_0 - s(\theta)). \\
&\quad + \int_0^\theta b'\left(A^{-1}(\tau + A(y_0))\right) exp\left(-\int_0^\tau a'\left(A^{-1}(l + A(y_0))\right)dl\right)d\tau\Big) \\
x(\theta) &= (t_0 - s(\theta))a(y_0) + b(y_0),
\end{aligned}
$$

where $A(r) = -\displaystyle\int_0^r \frac{1}{a(\tau)}d\tau$ is a notation.

4.2 A variational approach

In the second section of this chapter, we refer to the bidimensional case:

$$g(x, y) = ct. \tag{19}$$

and we solve (19) using a variational argument.

We introduce the Lagrangian:

$$L(x, u) = \sup_{y \in \mathbb{R}}\{uy - g(x, y)\}.$$

and we use the correspondence between the Hamiltonian and the associated convex Lagrangian, Barbu, Precupanu, 1978, [3]:

$$g(x, y) = \sup_{u \in \mathbb{R}}\{uy - L(x, u)\}.$$

Using this, we associate a minimization problem for (19):

$$\min_{q \in A} \int_0^T L(q(t), q'(t))dt. \tag{20}$$

We demonstrate that by solving problem (20) for the Lagrangian, we solve the implicit function problem (19). So, the implicit function problem can be solved by minimization, at least for some level lines.

For this demonstration, see Nicolai, 2015, [24].

5 Applications in optimization

In this chapter we use iterated Hamiltonian systems in the case of non-linear programming, with restrictions. We will solve the equality constraints and reduce the dimension of the minimization problem, Nicolai, 2016, [25]. This method can be applied in the critical case too and we quote Tiba, 2018, [34].

5.1 A problem in $\mathbb{R}^6$, Stuber et al., 2015, [32]

In the first section we consider a problem in $\mathbb{R}^6$, that was studied in M. D. Stuber, J. K. Scott, P. I. Barton, 2015, [32] and has applications in chemistry. We use the implicit parametrization method to obtain the solution of this problem:

We have $Z \subset \mathbb{R}^3$ and $P \subset \mathbb{R}^3$. Consider the objective function $f : Z \times P \to \mathbb{R}$:

$$
\begin{aligned}
f(z,p) = \sum_{j=1}^{3} \Bigg(& [a_j(p_j - c_j)]^2 + \sum_{i \neq j} a_j(p_i - c_i) \\
-5 & \left((j-1)(j-2)(z_2 - z_1) + \sum_{i=1}^{3} (-1)^{i+1} z_j \right) \Bigg)^2 ,
\end{aligned}
\tag{21}
$$

where a_i, c_i, $i = 1, 2, 3$ are fixed constants, given in Table 1 and the equality constraints are:

$$
\begin{aligned}
h_1(z,p) &= 10^{-9}\left(e^{38z_1} - 1\right) + p_1 z_1 - 1.6722 z_1 + 0.6689 z_3 - 8.0267 = 0, \\
h_2(z,p) &= 1.98 \cdot 10^{-9}\left(e^{38z_2} - 1\right) + 0.6622 z_1 + p_2 z_2 + 0.6622 z_3 + 4.0535 = 0, \\
h_3(z,p) &= 10^{-9}\left(e^{38z_3} - 1\right) + z_1 - z_2 + p_3 z_3 - 6 = 0.
\end{aligned}
\tag{22}
$$

	$i=1$	$i=2$	$i=3$
a_i	$37,3692$	$18,5805$	6.25
c_i	0.602	1.211	3.6

Table 1: constants

Using our method, we get better results than the ones obtained in M. D. Stuber, J. K. Scott, P. I. Barton, 2015, [32], but for that we extended the searching region, so there is no contradiction. See Nicolai, 2016, [25].

5.2 Problems in higher dimension

In this section we study problems in higher dimension. We give numerical examples for minimization problems in dimension 12 and 50, but this method can be extended for higher dimensions too. We also compare our results with the ones obtained with MultiStart routine from MATLAB. In every case, the implicit parametrization method offers us better results.

A very important observation is that this minimization method works very well for problems with many restrictions, because using them, we reduce the number of iterated Hamiltonian systems that need to be solved.

We now give an example from Nicolai, 2016, [25].

Example 1. In $\mathbb{R}^{12}$ we consider the problem:

$$\min f(x), \tag{23}$$

where

$$f(x) = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12}$$
$$or$$
$$f(x) = x_1^2 + 3x_2 + x_3 + x_4x_5 + x_6 + x_7^3 + x_8 + x_9 - x_{10}x_{11} + x_{12},$$

with ten equality constraints:

$$\begin{aligned}
F_1(x) &= x_1 - x_{11}x_{12} = 0,\\
F_2(x) &= 2x_1^2 - x_2 - x_{11}^2x_12 = 0,\\
F_3(x) &= x_1^2 - x_2 - x_3 + x_{11}x_{12} = 0,\\
F_4(x) &= 2x_1 - x_2 - x_3 + x_4 + x_{11}^2x_{12} = 0,\\
F_5(x) &= x_1^3 + x_2 - x_3 + 3x_4 - x_5 - 3x_{11}x_{12} = 0,\\
F_6(x) &= x_1^3 + x_2^2 + x_3 + x_5 - x_6 - x_{11}^2 - 2x_{12} = 0,\\
F_7(x) &= x_1 - x_2^3 - x_4 + x_5 - x_7 + x_{11}x_{12} = 0,\\
F_8(x) &= x_1 - x_3^3 - 2x_5 + x_7 - x_8 + x_{11}^2 + x_{12}^2 = 0,\\
F_9(x) &= -x_2 + x_3^3 - x_4 + 3x_6 - x_7 - x_9 - x_{11} + x_{12}^2 = 0,\\
F_{10}(x) &= x_1^3 + x_3^2 - x_4^2 + 2x_6 - x_7 - 3x_8 + x_9 - x_10 + x_{11}^3 - x_{12}^2 = 0,
\end{aligned} \tag{24}$$

We choose the initial condition $x^0 = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$ and using the implicit parametrization method, we get the following solution:

$\min f(x) = -1.5988 \cdot 10^{10}$, that is obtained for $x = [-2.4967 \cdot 10^{13}, 6.0715 \cdot 10^{18}, -2.4967 \cdot 10^{13}, -2.4967 \cdot 10^{13}, 2.4964 \cdot 10^{13}, 3998, -5.6136 \cdot 10^{-6}, 3.9960 \cdot 10^{6}, -7.9879 \cdot 10^{9}, -8.0039 \cdot 10^{9}, 1.2490 \cdot 10^{-16}, -1999]$.

For more numerical examples we recommend Nicolai, 2016, [25].

5.3 A polynomial / rational case

In this section, we present the Shekel problem, which was introduced in Shekel, 1971, [30] and studied in many papers, for example Ali, Khompatraporn, Zabinsky, 2005, [1], Lasserre, 2015, [20].

We consider the Shekel problem:

$$\min_{x} f(x) = -\sum_{i=1}^{n} \frac{1}{\sum_{j=1}^{m}(x_j - a_{ij})^2 + c_i}, \tag{25}$$

with $m = 4$, the constants a_{ij} şi c_i given and with the following inequality constraints:

$$0 \leq x_j \leq 10, \;\; j = 1, 2, \ldots, m\}.$$

This problem (and all of its variants) is studied for $n = 5, 7$ and $n = 10$ in the above mentioned papers (see also Dixon, Szegö, 1978, [11]). It can be extended to $n = 30$, and it is called *the Shekel Foxholes problem* (Bersini, Dorigo, Langerman, Seront, Gambardella, 1996, [4]).

In the next example we solve the Shekel problem with $n = 10$, for which we consider two extra equality constrains.

Example 2. Let the Shekel problem (25) with the constraints a_{ij} şi c_i given in Table 2 and with the restriction

$$F_1 = x_1x_2 - x_3x_4 - 4x_1 - 3.5x_2 + 4x_3 + 3.5x_4 = 0, \tag{26}$$
$$F_2 = 2x_2^2 - x_3^2 - x_4^2 = 0, \tag{27}$$
$$0 \leq x_j \leq 10, \;\; j \in \{1, 2, 3, 4\}, \tag{28}$$

We choose the initial condition $x^0 = [3.5, 3.5, 3.5, 3.5]$ and we observe that in x^0, both the equality constraints (26), (27) and the inequality constraints (28) are satisfied. We calculate the value of the Jacobian in the initial condition:

i	a_{ij}				c_i
	$j=1$	2	3	4	
1	4	4	4	4	0.1
2	1	1	1	1	0.2
3	8	8	8	8	0.2
4	6	6	6	6	0.4
5	3	7	3	7	0.4
6	2	9	2	9	0.6
7	5	5	3	3	0.3
8	8	1	8	1	0.7
9	6	2	6	2	0.5
10	7	3.6	7	3.6	0.5

Table 2: Constants Example 2

$$J = \frac{D(F_1, F_2)}{D(x_1, x_2)}(x^0) = 4x_2(x_2 - 4)/_{x^0} = -7 \neq 0, \tag{29}$$

and we denote by $A(x)$ the corresponding 3×3 non-singular submatrix of the Jacobian (29). So, $detA(x) = 4x_2(x_2 - 4)$. We solve the linear system:

$$v_l(x) \cdot \nabla F_s(x) = 0, \;\; s = 1, 2, \;\; l = 1, 2,$$

where the last two elements of $v_l(x)$ are the rows of the matrix $I_2 \cdot detA(x)$ and we obtain $v_l(x) \in \mathbb{R}^4$, $l = 1, 2$ in a neighborhood of x^0:

$$\begin{aligned} v_1(x) &= (4x_2x_4 - 16x_2 - 2x_1x_3 + 7x_3, 2x_2x_3 - 8x_3, 4x_2^2 - 16x_2, 0), \\ v_2(x) &= (-2x_1x_4 + 7x_4 + 4x_2x_3 - 14x_2, 2x_2x_4 - 8x_4, 0, 4x_2^2 - 16x_2). \end{aligned}$$

We form the following iterated differential systems, in which the right part is given by the vectors $v_l(x)$, $l = 1, 2$:

$$\begin{array}{ll} x_1'(t) = 4x_2(t)x_4(t) - 16x_2(t) - 2x_1(t)x_3(t) + 7x_3(t) & x_1(0) = 3.5 \\ x_2'(t) = 2x_2(t)x_3(t) - 8x_3(t) & x_2(0) = 3.5 \\ x_3'(t) = 4x_2^2(t) - 16x_2(t) & x_3(0) = 3.5 \\ x_4'(t) = 0 & x_4(0) = 3.5 \end{array}$$

$$\begin{aligned}
\dot{y}_1(s,t) &= -2y_1(s,t)y_4(s,t)+7y_4(s,t)+4y_2(s,t)y_3(s,t)-14y_2(s,t)\\
\dot{y}_2(s,t) &= 2y_2(s,t)y_4(s,t) - 8y_4(s,t),\\
\dot{y}_3(s,t) &= 0\\
\dot{y}_4(s,t) &= 4y_2^2(s,t) - 16y_2(s,t)
\end{aligned}$$

$$\begin{aligned}
y_1(0,t) &= x_1(t)\\
y_2(0,t) &= x_2(t)\\
y_3(0,t) &= x_3(t)\\
y_4(0,t) &= x_4(t)
\end{aligned}$$

We solve in order the iterated systems, using MATLAB by choosing the intervals of integration (for both systems) $[-10, 10]$ and the steps 0.001, respectively 0.01. From all of the obtained solutions, we keep just the ones that verify the inequality constraints (28). From all of this, we choose the one that offers the minimum value for the function.This way, we obtain the following result for the problem (25), (26), (27), (28):

$$f_{min} = -10.3707,\ x^\star = [3.9837, 3.9899, 3.9740, 4.0044],\ time \approx 10\ minutes,$$

with the restrictions satisfied

$$\begin{aligned}
&(F_1(x^\star)) = -0.0070,\\
&(F_2(x^\star)) = 0.0097,\\
&0 \le x_i^\star \le 10,\ i = 1, 2, 3, 4.
\end{aligned}$$

We see that the result is almost similar with the global minimum of the Sheckel problem, $min = 10.5319$, which is obtained for $x_{min} = [4, 4, 4, 4]$, see Ali, Khompatraporn, Zabinsky, 2015, [1], Dixon, Szegö, 1978, [11] (the difference is given by the approximation errors).

The implicit parametrization method offers a very efficient solution (the global solution) for the Shekel problem with some equality constraints added.

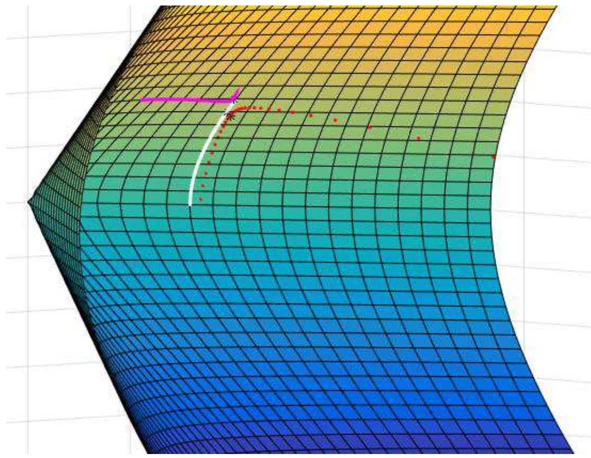

Figure 1: The evolution of admissible set at three different "moments": $x_1 = 3.5$ (white), $x_1 = 3.7$ (red), $x_1 = 3.9$ (magenta).

The same problem is studied in Lasserre, 2015, [20], p. 160. The author changes the inequality restrictions with:

$$\sum_{i=1}^{p} (x_i - 5)^2 \leq 60, \tag{30}$$

$p = 5$ sau $p = 10$.

We observe that the above calculated solution also satisfies restriction (30) and we obtain the same result.

If we consider x_1 a time variable, we can represent the evolution of admissible curves (Figure 1) (overlapped over the surface given by $F_2 = 0$).

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