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SOME REMARKS ON STATE CONSTRAINTS AND MIXED CONSTRAINTS *

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Dedicated to Professor Mihail Megan
on the occasion of his 70th anniversary

Abstract

We review some results on optimal control problems with both state and control constraints, or general mixed constraints, including certain recent developments. In the setting of state constrained control problems, we consider an approximation technique involving variational inequalities. The constraints may be automatically satisfied in this procedure. For control problems with mixed constraints, a relaxation of classical interiority assumptions is presented together with a recent approach based on implicit parametrization results and yielding global type algorithms.

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1 Introduction

This is a survey paper devoted to the subject of general constraints in optimal control theory. The subject is vast, difficult and of high interest. We shall concentrate on several directions of investigation, related to fundamental questions like approximation, necessary conditions, global algorithms. This choice is due to the personal experience of the author and it includes as well very recent progress in the field. Comparisons, comments on other methods and results are also indicated.

In the next section, we discuss the variational inequality approach to state constrained optimal control problems. Approximation and equivalence theorems are reviewed. A comparison with the penalization technique is provided. This and the next section are written in the setting of parabolic systems or for abstract evolution equations.

In Section 3, we continue the investigation of such problems, including the case of abstract mixed constraints and introducing a very weak constraint qualification condition that allows void interiors for the constraints sets. Bang-bang type results and applications in some economical models are also briefly mentioned.

The last section reports on very recent developments in global type approximation methods for optimal control problems governed by ordinary differential systems with general constraints: state, control, mixed, equality, inequality, abstract etc. The approach is based on the implicit parametrization approach, [21].

2 Approximation and equivalence

We consider $V, H, \mathcal{U}$ to be Hilbert spaces with dense and compact imbedding $V \subset H \subset V^*$ (the dual space of V - H is identified with its own dual space) and $A : V \to V^*$, $B : \mathcal{U} \to H$ to be linear bounded operators with the assumptions:

$$(Av, v)_{V^* \times V} \geq \omega |v|_V^2, \omega > 0, \forall\, v \in V, \tag{1}$$

$$(Ay, v)_{V^* \times V} = (y, Av)_{V \times V^*}, \forall\, y, v \in V. \tag{2}$$

The state constrained optimal control problem is defined by

$$\mathrm{Min}\{\mathrm{g(y) + h(u)}\}, \tag{3}$$

$$y' + Ay = Bu + f \text{ a.e. in } [0, T], \tag{4}$$

$$y(0) = y_0, \tag{5}$$

$$y(t) \in C, \ t \in [0, T], \tag{6}$$

Above, $C \subset H$ is a nonvoid, closed, convex subset, $y_0 \in C$, $Ay_0 \in H$, $f \in L^2(0, T; H)$, $g : L^2(0, T; H) \to R$ is convex, continuous, majorized from below by a constant and $h : L^2(0, T; \mathcal{U}) \to]-\infty, +\infty]$ is convex, proper, lower semicontinuous and coercive:

$$\lim_{|u|_{L^2(0,T;\mathcal{U})} \to \infty} h(u) = \infty. \tag{7}$$

Notice that control constraints $u \in U_{ad} \subset L^2(0, T; \mathcal{U})$, a convex closed subset, may be included in the definition of h by adding the corresponding indicator function.

For any $u \in L^2(0, T; H)$, the equation (4), (5) has a unique solution $y \in C(0, T; V)$, $y' \in L^2(0, T; H)$ due to (1), (2). Under the usual admissibility hypothesis the control problem (3) - (6) has an optimal pair $[y^*, u^*]$ due to (7) and unique if strict convexity is assumed for the cost functional (3).

If $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, the usual Sobolev spaces in the smooth bounded domain $\Omega \subset R^d$, and A is an elliptic operator in $H_0^1(\Omega)$, then the equation (4), (5) becomes a parabolic problem.

One variant of the variational inequality approximation technique is to associate with the constrained control problem (3) - (6), the approximate problem without state constraints:

$$\text{Min}\{g(y) + h(u) + \frac{1}{2}|w|^2_{L^2(0,T;V^*)}\}, \tag{8}$$

$$y' + Ay + \varepsilon\omega = Bu + f, \varepsilon > 0, w \in \partial\varphi(y), \tag{9}$$

$$y(0) = y_0, \tag{10}$$

where $\varphi : V \to]-\infty, +\infty]$ is convex, lower semicontinuous, proper

$$\varphi(v) = \begin{cases} 0 & v \in C \cap V, \\ +\infty & \text{otherwise.} \end{cases} \tag{11}$$

The variational inequality (9), (10) has a unique solution $y \in C(0,T;H) \cap L^2(0,T;V)$, $y' \in L^2(0,T;H)$ by standard existence results in the literature [2]. Moreover $w \in L^2(0,T;V^*)$ as well (related to the section of $\partial\varphi(y)$ occuring in (9)).

Using standard techniques involving minimizing sequences [19], it is possible to show that the unconstrained control problem (8) - (10) has at least one optimal pair $[y_\varepsilon, u_\varepsilon]$ for any $\varepsilon > 0$. The justification of this approach is given by the following result [12]:

Theorem 1 *If h is strictly convex and superquadratic and $\mathcal{U} = L^2(\Omega)$, then*

$$u_\varepsilon \to u^*, \text{ strongly in } L^2(0,T;H),$$

$$y_\varepsilon \to y^*, \text{ strongly in } C(0,T;H).$$

If we denote by y^ε the solution of (4), (5) corresponding to u_ε, then

$$\operatorname{dist}(y^\varepsilon, C \cap V)_{C(0,T;H)\cap L^2(0,T;V)} \le k\varepsilon, \tag{12}$$

where $k > 0$ is independent of $\varepsilon > 0$.

This shows the suboptimal character of the control u_ε, including an explicit uniform estimate of the possible violation of the state constraint (6). Theorem 1 can be strengthened to include a regularization of the nonlinear operator $\partial\varphi$ that appears both in the cost (8) and in the state equation (9).

Slightly weaker estimates as in (12) may be obtained as well. For the regularized problems, usual gradient methods may be applied on numerical results are indicated in [12]. The variational inequality approach is a refinement of the penalization method with better estimates.

Assume now that $B : \mathcal{U} \to V^*$ linear bounded, $f \in L^2(0,T;V^*)$ and $C \subset V$. Then, $y \in C(0,T;H) \cap L^2(0,T;V)$, $y' \in L^2(0,T;V^*)$ as defined in (4), (5). Let $B^* : V \to \mathcal{U}^*$ be the adjoint operator and $\widetilde{C} = \{v \in V, B^*v \in B^*(C)\}$. We introduce the unconstrained problem

$$\operatorname{Min}\{g(y) + h(u - w) + \frac{1}{2}|w|^2_{L^2(0,T;\mathcal{U})}\}, \tag{13}$$

$$y' + Ay + Bw = Bu + f, \ w \in \partial\psi(B^*y), \tag{14}$$

$$y(0) = y_0, \tag{15}$$

where ψ is the indicator function of $B^*(C)$ in $\mathcal{U}^*$. Under certain condition on $\mathrm{dom}(\psi) \cap \mathrm{range}(B^*)$ of interiority type [18], [3], the equation (14) can be rewritten in the form

$$y' + Ay + \partial\widetilde{\phi}(y) \ni Bu + f, \tag{16}$$

where $\widetilde{\phi}$ is the indicator of $\widetilde{C}$ in V.

Theorem 2 *The control problems (13) - (15) and (3) - (6) are equivalent in the sense that they have the same optimal values and optimal pairs.*

This section is based on [12] and [5] where more algorithms and equivalence results and applications to bang-bang properties may be found.

3 Interiority assumptions and generalized optimality conditions

We discuss here a more general optimal control problem involving abstract mixed constraints:

$$\mathrm{Min}\{L(y,u) + l(y(T))\}, \tag{17}$$

$$y'(t) + A(t)y(t) = Bu(t) + f(t) \text{ a.e. in } [0,T], \tag{18}$$

$$[y,u] \in D \subset [L^2(0,T;V) \cap W^{1,2}(0,T;V^*)] \times L^2(0,T;\mathcal{U}). \tag{19}$$

Here D is a convex closed nonvoid subset, $f \in L^2(0,T;V^*)$, $L : L^2(0,T;H\times \mathcal{U}) \to R$, $l : H \to R$ are convex, continuous mappings, with the coercivity property:

$$L(y,u) \geq c_1|u|^2_{L^2(0,T;\mathcal{U})} - c_2, c_i > 0 \text{ constants.} \tag{20}$$

The family of operators $A(t)$ is V^* - measurable on $]0,T[$ and satisfies conditions like (1), (2) with uniform in $t \in [0,T]$ constants. The solution of (18), with initial condition $y(0) = y_0 \in H$ is unique in $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$.

Under the admissibility condition, hypothesis (20) ensures the existence of at least one optimal pair $[y^*,u^*] \in D$ for the problem (17) - (19).

To obtain the optimality conditions for the problem (17) - (19), it is usual [3] to impose Slater type assumptions

$$\exists\ [\bar{y}, \bar{u}] \text{ feasible } : \bar{y} \in \text{int}\{y \in C(0,T;H); [y, \bar{u}] \in D\}. \tag{21}$$

In the papers [4], [22], [6] related problems and applications are studied. We denote the operator

$$\begin{gathered} \forall\ [y,u] \in [L^2(0,T;V) \cap W^{1,2}(0,T;V^*)] \times L^2(0,T;V), \\ T(y,u) = y' + A(t)y - Bu - f \in L^2(0,T;V^*). \end{gathered} \tag{22}$$

We impose the following weak constraint qualification

$$\begin{gathered} \exists\ \mathcal{M} \in D, \text{ bounded in } C(0,T;H) \times L^2(0,T;\mathcal{U}) : \\ 0 \in \text{ int } T(\mathcal{M}) \text{ in } L^2(0,T;V^*). \end{gathered} \tag{23}$$

It is easy to show that (21), (22) give (23). One can also infer that (23) is, as well, a consequence of

$$\exists\ [\bar{y}, \bar{u}] \text{ feasible } : \bar{u} \in \text{ int}\{u \in L^2(0,T;V); [\bar{y}, u] \in D\}. \tag{24}$$

Clearly (21) may not be valid if (24) is imposed. That is, constraint qualification (23) is strictly weaker than the slater condition. One can compare (23) with the Zowe and Kurcyusz constraint qualification [23] in mathematical programming.

By using an adapted regularization and penalization of (17) combined with the penalization of (18), rather delicate duality-type estimates give the following generalized optimality system [4]:

Theorem 3 *If the pair $[y^*, u^*]$ is optimal for the problem (17) - (19), then:*

$$\int_0^T (y^{*'} - y' + Ay^* - Ay, p^* + r^*)_{V^* \times V} dt \leq 0, \tag{25}$$

$$\langle w_2, u^* - u \rangle_{L^2(0,T;\mathcal{U})} - \int_0^T \langle u^* - u, B^* J^{-1}(r^*) \rangle_{\mathcal{U}} dt \leq 0, \tag{26}$$

for any $[y,u]$ such that $[y,u^*] \in D$ and $[y^*,u] \in D$.

Moreover, summing (25) and (26) is valid for any $[y,u] \in D$ and it is also sufficient for the optimality of $[y^*,u^*]$.

Here, p^* is the solution of the adjoint system

$$-p^{*'} + A^* p^* = w_1, \tag{27}$$

$$p^*(T) = w, \tag{28}$$

where $w \in \partial l(y^*(T))$, $[w_1, w_2] \in \partial L(y^*, u^*]$ and $J : V \to V^*$ is the canonical isomorphism, $r^* \in L^2(0,T;V)$ is the weak limit (on a subsequence) of

$$r_\varepsilon = \frac{1}{\varepsilon} J^{-1}(y'_\varepsilon + A(t)y_\varepsilon - Bu_\varepsilon - f)$$

with $[y_\varepsilon, u_\varepsilon]$ being the unique optimal pair of the approximating regularized/penalized optimal control problem. See [4] for more details.

Remark 1 *The form (25), (26) decouples the adjoint system (27), (28) from the constraints. In case, $D = \mathcal{K} \times \mathcal{U}_{ad}$ (separate state and control constraints), one can easily reobtain from (25) - (28) the usual form of the optimality conditions, [3].*

We briefly comment now on an example from [4] that shows that even in the case of separate constraints, their interior may be void, while hypothesis (22), (23) is satisfied. We consider the following optimal control problem governed by a parabolic equation:

$$\text{Min}\{\frac{1}{2}\int_Q (y - z_d)^2 dx + \frac{N}{2}\int_Q u^2 dx\}, \tag{29}$$

$$\frac{\partial y}{\partial t} - \Delta y = f + u \text{ in } Q = \Omega \times]0, T[, \tag{30}$$

$$y(x,t) = 0 \text{ on } \Sigma = \partial\Omega \times [0,T], \tag{31}$$

$$y(x,0) = y_0(x) \text{ in } \Omega, \tag{32}$$

$$e(x,t) \le y(x,t) \le g(x,t) \text{ a.e. in } Q, \tag{33}$$

$$a(x,t) \le u(x,t) \le b(x,t) \text{ a.e. in } Q, \tag{34}$$

where $\Omega \subset R^d$ is a smooth bounded domain, $z_d \in L^2(Q)$, $N \ge 0$, $y_0 \in L^2(\Omega)$, f, a, b are in $L^\infty(Q)$ and e, g are in $C(\bar{Q})$. From (33), (34) one can immediately infer the form of D from (19).

We ask the "rich" admissibility hypothesis:

$\exists\ \alpha > 0, \exists\ \widetilde{u}$ satisfying (34) such that if Y denotes the operator $u \to y$ defined by (30) - (32), we have:

$$e \leq Y(\widetilde{u} - \alpha) \leq Y(\widetilde{u} + \alpha) \leq g \text{ a.e. in } Q. \tag{35}$$

Relation (35) is not an interiority condition since, we may have $e(x,t) = g(x,t)$ in certain points, for instance $e(x,t) = g(x,t) = 0$ on $\partial\Omega \times [0,T]$. Moreover $\widetilde{u}+\alpha$, $\widetilde{u}-\alpha$ need not satisfy (34) and we may as well have $a(x,t) = b(x,t)$ on some subset.

Taking the spaces $V = H_0^1(\Omega)$, $H = \mathcal{U} = L^2(\Omega)$ and the operators $A(t) = -\Delta$, $B : H \to V^*$, $B = i$, the canonical injection and the cost $L(y,u)$ as given in (29), while $l = 0$, one can put the example (29) - (34) in the abstract form (17) - (19).

The condition (23) may be checked with

$$\mathcal{M} = \text{conv}\{[y_\xi, u_\xi]; \xi \in L^\infty(Q), \|\xi\|_{L^\infty(Q)} = 1, y_\xi = y(u_\xi), u_\xi = \widetilde{u} + \alpha\xi\}$$

(notice that the space $C(0,T;H) \times L^2(0,T;\mathcal{U})$ is replaced by $L^\infty(Q)$ in this example). The arguments are similar and take advantage that we work in a functional setting and we can use comparison theorems.

In the work [6] other examples of mixed constraints, in connection with an optimal investment problem, are discussed. The setting is similar with the above problem governed by parabolic partial differential equations:

$$\frac{1}{2}\int_\Omega y(x,t)^2 dx \leq C(u)(t), t \in [0,T],$$

where $C(\cdot) : \mathcal{U} \to L^1(0,T)$ is some given operator.

$$0 \leq u(x,t) \leq Cy(x,t) \text{ a.e. in } Q.$$

One can establish generalized bang-bang properties for such applications, that give almost complete information on the optimal pairs. Further applications to error estimates in discretization procedures for state constrained optimal control problems are reported in [22].

4 An implicit parametrization approach in the ODE setting, with equality constraints

We briefly review now the Hamiltonian approach to implicit systems, according to [14], [15], [20], [21], [13]. Then, we show how to apply it to optimal control problems governed by ordinary differential equations, with equality mixed constraints.

We underline that inequality constraints, abstract constraints, separated constraints are also allowed under our approach, as we shall point out later. This is a generalization of the methods developed for nonlinear programming problems, from Tiba [21].

In the Euclidean space R^d, we consider the general implicit functions system:

$$\begin{aligned} &F_1(x_1,\dots,x_d)=0,\\ &F_2(x_1,\dots,x_d)=0,\\ &\quad\dots\dots\dots\dots\\ &F_l(x_1,\dots,x_d)=0, \end{aligned} \tag{36}$$

where $1 \le l \le d-1$ and $F_j \in C^1(\Omega)$, $F_j(x^0)=0$, $j=\overline{1,l}$, $x_0 \in \Omega \subset R^d$, given bounded domain.

We assume the standard independence assumption

$$\frac{D(F_1,F_2,\dots,F_l)}{D(x_1,x_2,\dots,x_l)} \neq 0 \text{ in } x^0, \tag{37}$$

however this hypothesis can be dropped and the notion of generalized solution as introduced in Tiba [20], [21] can be used for the solving of (36). The fact that the "first" independent variables $x_1,\dots,x_l$ appear in (37) can always be obtained by renumbering. Notice that condition (37) remains valid in a neighbourhood of x^0, that we denote by $V \subset \Omega$.

We introduce in V the undetermined system of linear algebraic equations

$$v(x)\cdot\nabla F_j(x)=0, j=\overline{1,l} \tag{38}$$

and we shall use $d-l$ solutions of (38) obtained by fixing successively the last $d-l$ components of the vector $v(x)\in R^d$ to be the rows of the identity matrix in R^{d-l} multiplied by $\Delta(x)=\det A(x)\neq 0$ (the determinant appearing in (37)). In this way we obtain $d-l$ independent solutions of (38) denoted by $v_1(x),\dots,v_{d-l}(x)$, in some arbitrary order.

Their first l components are obtained from (38) by inverting $A(x)$, $x\in V$, due to (37).

The vector fields $v_k(x)$, $k=\overline{1,d-l}$ are also continuous in V since $F_j, j=\overline{1,l}$ are of class $C^1(\Omega)$. This choice of $\{v_k\}_{k=\overline{1,d-l}}$ is not the unique useful choice, [20], [21], [13]. We associate to them the following iterated type Hamiltonian system of differential equations (weakly coupled just via the initial conditions - that's why we call it iterated):

$$\begin{aligned} \frac{\partial y_1(t_1)}{\partial t_1} &= v_1(y_1(t_1)), t_1 \in I_1 \subset R, \\ y_1(0) &= x^0, \end{aligned} \tag{39}$$

$$\begin{aligned} \frac{\partial y_2(t_1,t_2)}{\partial t_2} &= v_2(y_2(t_1,t_2)), t_2 \in I_2(t_1) \subset R, \\ y_2(t_1,0) &= y_1(t_1), \end{aligned} \tag{40}$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\begin{aligned} \frac{\partial y_{d-l}(t_1,t_2,\dots,t_{d-l})}{\partial t_{d-l}} &= v_{d-l}(y_{d-l}(t_1,t_2,\dots,t_{d-l})), t_{d-l} \\ &\in I_{d-l}(t_1,t_2,\dots,t_{d-l-1}), \\ y_{d-l}(t_1,t_2,\dots,t_{d-l-1},0) &= y_{d-l-1}(t_1,t_2,\dots,t_{d-l-1}). \end{aligned} \tag{41}$$

Although partial differential notations are used in (39) - (41), each of the above subsystems may be interpreted as an ordinary differential system since just one derivative appears. The Hamiltonian character of (39) - (41) will be obvious from their properties listed in what follows, and from the example. Existence is valid by the Peano theorem.

Theorem 4 *Under assumption (37), if $l = d-1$, the system (39) - (41) has the uniqueness property.*

If $1 \leq d \leq d-2$, every subsystem of (39) - (41) has the uniqueness property. Moreover, the intervals $I_j(t_1,t_2,\dots,t_{j-1})$, $j = \overline{1,d-l}$ may be choosen independent of the previous parameters.

The classical conservation property of Hamiltonian systems is valid in this general setting:

Theorem 5 *For every $k = \overline{1,l}$, $j = \overline{1,d-l}$, we have:*

$$F_k(y_j(t_1,t_2,\dots,t_j)) = 0, \forall\ (t_1,t_2,\dots,t_j) \in I_1 \times I_2 \times \dots \times I_j,$$

where $I_1,\dots,I_j$ denote the existence intervals, without the dependence on the previous parameters, that is not necessary in fact.

The most important property of the iterated Hamiltonian type system (39) - (41) is the fact that it provides, in a constructive manner and in the most general situation, a parametrization of the manifold defined by the implicit system (36), under condition (37). This is called the implicit parametrization property:

Theorem 6 *If $F_k \in C^1(\Omega)$, $k = \overline{1,l}$, with the independence property 37 and $I_j, j = \overline{1, d-l}$ are sufficiently small, then the mapping*

$$y_{d-l} : I_1 \times I_2 \times \cdots \times I_{d-l} \to R^d$$

is C^1 in all its variables, regular and one-to-one on its image.

Moreover, the classical implicit function theorem follows as a special case of the above approach if in the solution of the algebraic system (38), the last $d - l$ components of the vector $v(x) \in R^d$, $x \in V$, are choosen to be exactly the rows of the identity matrix in R^{d-l}. This is a constructive approach to implicit functions, [21]. We also obtain precise estimates of the existence intervals $I_j, j = \overline{1, d-l}$. For instance, if $V = B(x_0, r)$ with $r > 0$ determined by (37), then we may choose $I_j = \left[-\frac{r}{(d-l)M}, \frac{r}{(d-l)M}\right]$, where $M > 0$ is the maximum $|v_j|_{C(\bar{V})}$, $j = \overline{1, d-l}$.

As already mentioned, the hypothesis (37) can be dropped and a constructive notion of generalized solution of (36), using the Hausdorff - Pompeiu convergence may be defined.

This is done in [20], [21] where basic properties are also proved.

We introduce now the optimal control problem with equality mixed constraints:

$$\text{Min}\{l(x(0), x(1))\}, \tag{42}$$

$$x'(t) = f(t, x(t), u(t)), t \in [0, 1], \tag{43}$$

$$h(x(t), u(t)) = 0, t \in [0, 1]. \tag{44}$$

Above, $l : R^d \times R^d \to R$, $f : [0, 1] \times R^d \times R^m \to R^d$, $h : R^d \times R^m \to R^s$, $s \geq m$, $d + m - 1 \geq s$, are given mappings and $x : [0, 1] \to R^d$ is the state variable, $u : [0, 1] \to R^m$ is the control unknown. Such systems are recently studied in a more general implicit form by Maria do Rosario de Pinho [16], [17], Clarke and Pinho [8], Frankowska [10] from the point of view concerning the maximum principle and under weak differentiability assumptions. The formulation (42) - (44) is of Mayer type and the conditions (43), (44) give a DAE system.

Initial conditions may be imposed in (43), but it is not necessary now. If inequality constraints are added to (42) - (44), the classical procedure is to introduce supplementary control variables in order to transform them in equality constraints. Our procedure is different, without increasing the dimension of the control space and applies as well to abstract constraints, separated constraints.

As general assumptions, we shall require l continous, f locally Lipschitzian in (x, u) and measurable in t, h of class C^1 and there is a point $(x^0, u^0) \in R^d \times R^m$ such that

$$h(x^0, u^0) = 0 \text{ and } \nabla h(x^0, u^0) \text{ of maximal rank.} \tag{45}$$

Notice that, in this setting, some of the constraints (44) may be separate state constraints or control constraints and the remaining constraints may be of mixed type.

Under hypothesis (45), one can use Thm. 4 - 6 to obtain a constructive parametric description of the admissible manifold for (44), denoted by $A \subset R^d \times R^m$. This is not the admissibility set for the control problem (42) - (44). However any admissible state-control trajectory should satisfy

$$(x(t), u(t)) \in A, t \in [0, T]. \tag{46}$$

Here and in the sequel we shall assume that the admissible controls $u(t) \in W^{1,2}(0, T; R^m)$, consequently (46) makes sense by regularity properties for (43). We also recall that the regularity results for the optimal pairs Clarke [7], Fleming and Rishel [9] allow to restrict the search for admissible pairs by such regularity conditions.

Remark 2 *This setting can be extended to weaker hypotheses on s, m and the properties of h.*

In the standard terminology for DAE system, relations (43), (44) are semi-explicit of index one. Taking into account (44), (46) and differentiating once, we get:

$$\nabla_x h(x(t), u(t)) f(t, x(t), u(t)) + \nabla_u h(x(t), u(t), u'(t)) = 0. \tag{47}$$

If $\nabla_u h(\cdot, \cdot)$ is invertible on A, then (47) may be put in an explicit form, as an ODE for the control vector $u(t) \in R^m$.

The important observation is that any point in A provides a consistent initial condition for the differential system (43), (47). This system gives a characterization of the admissible state-control trajectories. It is elementary to show:

Proposition 1 *Any trajectory of (43), (47), starting from a point in A, remains in A and is in $W^{1,2}((0, s); R^d \times R^m)$.*

Here, $(0, s)$ is the local existence interval (depending on the initial condition). In control theory, taking into account the form of the cost functional (42), we also require the existence of global solutions, in $[0,1]$, that has to be checked in each application.

We notice that the set of discretization points in A generated by (39) - (41), denoted by $\underset{n\in N}{\cup} A_n$, is dense in A when the discretization of $I_1 \times I_2 \times \ldots \times I_{d-l}$ is finer and finer.

We have, for the terminal set $T = \{x(1); [x(0), u(0)] \in A\}$:

Proposition 2 *Under global existence for the system (43), (47), the discretized terminal set* $\underset{n\in N}{\cup} T_n = \{x(1); [x(0), u(0)] \in \underset{n\in N}{\cup} A_n\}$ *is dense in* T.

This is a consequence of continuity results with respect to initial conditions, Barbu [1], Hartman [11] since f is locally Lipschitz in (x, u) and similar conditions are imposed on $\nabla h(\cdot, \cdot)$.

We add now to the problem (42) - (44) more constraints:

$$q_r(x(t), u(t)) \le 0, r = \overline{1, Q}, t \in [0,1], \tag{48}$$

$$(x(t), u(t)) \in C(t), t \in [0,1], \tag{49}$$

where $C(t)$ is some closed nonvoid subset, for any $t \in [0,1]$ and assume that the admissible set for (43), (44), (48), (49) is not empty. The following algorithm is taken into account:

Algorithm 4.7

1) Fix $n = 1$ and choose some discretization of $I_1 \times I_2 \times \ldots \times I_{d-l}$, a tolerance parameter δ etc.

2) Compute A_n via (39) - (41), the corresponding discretization of A.

3) Compute via (43), (47) the trajectories $[x(t), u(t)]$, with initial conditions in A_n.

They automatically satisfy (44).

4) Check the conditions (48), (49) for all the trajectories defined in STEP 3 (in the discretization points). This gives the set of admissible discrete trajectories $\mathcal{O}_n$.

5) Compute $l(x(0), x(1))$ for all $[x, u] \in \mathcal{O}_n$ and find the optimal solutions (which may be not unique) and the optimal cost L_n.

6) If $|L_n - L_{n-1}| < \delta$, then STOP!

Otherwise $n := n + 1$ and GO TO STEP 2.

This algorithm has a global character, although the problem (42) - (44), (48), (49) is strongly nonconvex. It is enough to have just one equality constraint (44) (or one inequality constraint (48)) in order for the **Algorithm 4.7** to work. It is also worth to mention, that it is advantageous to have many equality constraints: the dimension of (39) - (41) is $d + m - s$ in this case. The dimension of the problem is given by the application. Usual routines in Matlab solve such ODE systems in several seconds, even for d "big".

The convergence of **Algorithm 4.7** is ensured by Prop. 2 and the continuity of $l(\cdot, \cdot)$. Other types of stopping tests in STEP 6 may be taken into account as well.

We close this section with the following example

Example 1 *We take $d = s = m = 1$ in (42) - (44), with hypothesis (45) and other conditions mentioned above. Then, the constraint (44) gives a curve in R^2, with coordinates (y, v). Its parametric representation can be obtained by the simplest Hamiltonian system:*

$$\begin{aligned} \dot{y}(s) &= -\frac{\partial h}{\partial u}(y(s), v(s)), s \in I \\ \dot{v}(s) &= -\frac{\partial h}{\partial x}(y(s), v(s)), s \in I \end{aligned} \tag{50}$$

with initial condition

$$y(0) = x^0, \quad v(0) = u^0. \tag{51}$$

Any admissible trajectory $(x(t), u(t))$, $t \in [0, 1]$, should lie on the curve defined by (50), (51) and is, in fact, completely determined by its initial conditions. The admissibility equation (47) for u has the form

$$\begin{aligned} u'(t(s)) &= -\frac{\frac{\partial h}{\partial x}(x(t,s),u(t,s))}{\frac{\partial h}{\partial u}(x(t,s),u(t,s))} f(t, x(t, s), u(t, s)), t \in [0, 1] \\ x(0, s) &= y(s), \quad u(0, s) = v(s) \end{aligned} \tag{52}$$

(with obvious notations for the derivatives in s, respectively t) and should be solved together with (43).

If $l(a, b) = (a-4)^2 + (b-15)^2$ and $f(t, x, u) = x - 5u + 10t + 2$, $h(x, u) = x - 5u - 4$, then $x(t) = 5t^2 + 6t + 4$, $u(t) = t^2 + 1.2t$ give an optimal pair.

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