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A CLASS OF FUNCTIONAL-INTEGRAL EQUATIONS VIA PICARD OPERATOR TECHNIQUE *

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Dedicated to Professor Mihail Megan
on the occasion of his 70th anniversary

Abstract

Let $\mathbb{B}$ be a Banach space, $\alpha \leq a < b \leq \beta$, $K \in C([\alpha,\beta]^2 \times \mathbb{B}^2, \mathbb{B})$, $g \in C([\alpha,\beta], \mathbb{B})$ and $h \in C([\alpha,\beta],[\alpha,\beta])$. In this paper, using the Picard operator technique, we will study, in $C([\alpha,\beta],\mathbb{B})$, the following integral equation

$$x(t) = \int_a^b K(t,s,x(s),x(h(s)))ds,\ t \in [\alpha,\beta].$$

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1 Introduction and Preliminaries

Let X be a nonempty set and $A : X \to X$ be a given operator. Then, we denote by $F_A := \{x \in X : \ x = A(x)\}$ the fixed point set for A and by $A^n = A \circ \cdots \circ A$ ($n \in \mathbb{N}^*$) the iterates of A.

Definition 1 *Let (X, d) be a metric space and let $A : X \to X$ be an operator. By definition, the operator A is said to be:*

(i) a Picard operator if $F_A = \{x^*\}$ and $A^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.

(ii) a c-Picard operator if A is a Picard operator, $c > 0$ and

$$d(x, x^*) \leq cd(x, A(x)), \ \textit{for all } x \in X.$$

For example, if (X, d) is a complete metric space and $A : X \to X$ is a k-contraction (i.e., $k \in (0, 1)$ and the following relation holds

$$d\left(A\left(x\right), A\left(y\right)\right) \leq kd\left(x, y\right), \ \text{for all } x, y \in X),$$

then A is a c-Picard operator with $c = \frac{1}{1-k}$. For the Picard operator theory see [11], [18], [12], [18], $\cdots$

The following result is essential in our approach.

Theorem 1 (Saturated principle of contraction [17]) *Let (X, d) be a complete metric space and $f : X \to X$ be an k-contraction. Then, the following conclusions hold:*

(i) *There exists $x^* \in X$ such that,*

$$F_{f^n} = \{x^*\}, \ \forall \ n \in \mathbb{N}.$$

(ii) *For all $x \in X$, $f^n(x) \to x^*$ as $n \to \infty$.*

(iii) *$d(x, x^*) \leq \psi(d(x, f(x)))$, for all $x \in X$, where $\psi(t) = \frac{t}{1-k}$, $t \geq 0$.*

(iv) *If $\{y_n\}_{n \in \mathbb{N}}$ is a sequence in X such that*

$$d(y_n, f(y_n)) \to 0 \ \textit{as } n \to \infty,$$

then, $y_n \to x^*$ as $n \to \infty$.

(v) *If $\{y_n\}_{n \in \mathbb{N}}$ is a sequence in X such that*

$$d(y_{n+1}, f(y_n)) \to 0 \ \textit{as } n \to \infty,$$

then, $y_n \to x^*$ as $n \to \infty$.

(vi) If $Y \subset X$ is a closed subset such that $f(Y) \subset Y$, then $x^* \in Y$. Moreover, if in addition Y is bounded, then

$$\bigcap_{n\in\mathbb{N}} f^n(Y) = \{x^*\}.$$

By the above result, several other properties of the fixed point equation can be deduced. For more details on the above approach see [17].

2 Existence and Uniqueness

Let $(\mathbb{B}, \|\cdot\|)$ be a Banach space, $\alpha.\beta, a, b$ be real numbers such that $\alpha \leq a < b \leq \beta$, $K \in C([\alpha,\beta]^2 \times \mathbb{B}^2, \mathbb{B}), g \in C([\alpha,\beta], \mathbb{B})$ and $h \in C([\alpha,\beta],[\alpha,\beta])$. In this paper, using the Picard operator technique, we will study, in $C([\alpha,\beta],\mathbb{B})$, the following integral equation

$$x(t) = \int_a^b K(t,s,x(s),x(h(s))ds + g(t),\ t \in [\alpha,\beta]. \tag{1}$$

We have the following existence and uniqueness result, which extends a theorem given in [8].

Theorem 2 *Let us consider the equation (1) under the above assumptions on K, g, h. We, additionally, suppose:*

(i) there exist $L_1, L_2 > 0$ such that

$$\|K(t,s,,u_1,v_1) - K(t,s,u_2,v_2)\| \leq L_1\|u_1 - v_1\| + L_2\|u_2 - v_2\|,$$

for each $t,s \in [\alpha,\beta], u_1, v_1, u_2, v_2 \in \mathbb{B}$.

(ii) $(L_1 + L_2)(b-a) < 1$.

Then, we have the following conclusions:

(a) the equation (1) has in $C([\alpha,\beta],\mathbb{B})$ a unique solution x^*;

(b) for all $x_0 \in C([\alpha,\beta],\mathbb{B})$, the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_{n+1}(t) := \int_a^b K(t,s,x_n(h(s)))ds + g(t)$$

converges uniformly on $[\alpha,\beta]$ to x^*.

(c) if $(y_n)_{n\in\mathbb{N}}$ is a sequence in $C([\alpha,\beta],\mathbb{B})$ such that the sequence $(z_n)_{n\in\mathbb{N}}$ defined by

$$z_n(t) := y_n(t) - \int_a^b K(t,s,y_n(s),y_n(h(s)))ds - g(t), t \in [\alpha,\beta]$$

converges uniformly to 0 *on* $[\alpha.\beta]$*, then* $(y_n)_{n\in\mathbb{N}}$ *converges uniformly on* $[\alpha,\beta]$ *to* x^* *as* $n\to\infty$*.*

(d) if $(u_n)_{n\in\mathbb{N}}$ *is a sequence in* $C([\alpha,\beta],\mathbb{B})$ *such that the sequence* $(v_n)_{n\in\mathbb{N}}$ *defined by*

$$v_n(t) := u_{n+1}(t) - \int_a^b K(t,s,u_n(s),u_n(h(s)))ds - g(t), t\in[\alpha,\beta]$$

converges uniformly to 0 *on* $[\alpha.\beta]$*, then* $(u_n)_{n\in\mathbb{N}}$ *converges uniformly on* $[\alpha,\beta]$ *to* x^* *as* $n\to\infty$*.*

Proof. Let us consider the operator $A : C([\alpha,\beta],\mathbb{B}) \to C([\alpha,\beta],\mathbb{B})$ defined by

$$A_{a,b}x(t) := \int_a^b K(t,s,x(s),x(h(s)))ds + g(t),\ t\in[\alpha,\beta].$$

It is obvious that the solution set of the equation (1) coincides with the fixed point set $F_{A_{a,b}}$ of the above mentioned operator.

From (i) and (ii) we can prove that $A_{a,b}$ is a $(L_1+L_2)(b-a)$-contraction on the Banach space $(C([\alpha,\beta],\mathbb{B}),\|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the usual supremum norm given by $\|x\|_\infty := \max\limits_{t\in[\alpha,\beta]}\|x(t)\|$. So, the proof follows by the Saturated Principle of Contraction. $\square$

Remark 1 *1) By the Saturated Principle of Contraction (see Theorem 1 and [17]) we also get that*

$$\|x-x^*\|_\infty \le \frac{1}{1-(L_1+L_2)(b-a)}\|x-A_{a,b}x\|_\infty,\ \textit{for all } x\in C([\alpha,\beta],\mathbb{B}).$$

This means that $A_{a,b}$ *is a* $\frac{1}{1-(L_1+L_2)(b-a)}$*-Picard operator.*

2) If $\mathbb{B}:=\mathbb{R}^m$ *or* $\mathbb{B}:=\mathbb{C}^m$*, Theorem 2 gives an existence and uniqueness result for a system of integral equations. The only difference is the fact that, if we are working with an* $\mathbb{R}^m_+$*-norm on* $C([\alpha,\beta],\mathbb{K}^m)$ *(where* $\mathbb{K}=\mathbb{R}$ *or* $\mathbb{C}$*), then the result follows by applying Perov's fixed point theorem, see for example [8].*

3) If $\mathbb{B}:=\ell^p(\mathbb{K})$ *(where* $1<p\le\infty$*), Theorem 2 gives an existence and uniqueness result for an infinite system of integral equations.*

4) For the fixed point technique in the theory of integral and functional-integral equations see [5], [9], [4], [7], [3], [6], [1], [8], [10], [15], $\cdots$

3 Data dependence with respect to K and g

Let us consider now the following equation

$$x(t) = \int_a^b \tilde{K}(t, s, x(s), x(h(s))ds + \tilde{g}(t),\ t \in [\alpha, \beta], \tag{2}$$

where $\tilde{K} \in C([\alpha, \beta]^2 \times \mathbb{B}^2, \mathbb{B})$ and $\tilde{g} \in C([\alpha, \beta], \mathbb{B})$.

Let us suppose that there exist $\eta_1, \eta_2 > 0$ such that

$$\|K(t, s, u, v) - \tilde{K}(t, s, u, v)\| \le \eta_1,\ \text{for all } t, s \in [\alpha, \beta] \text{ and } u, v \in \mathbb{B} \tag{3}$$

and

$$\|g(t) - \tilde{g}(t)\| \le \eta_2,\ \text{for all } t \in [\alpha, \beta]. \tag{4}$$

Then, by Remark 1 and Theorem 2, we have the following data dependence result.

Theorem 3 *Let us consider the equation (1) satisfying the hypotheses of Theorem 2. We also consider equation (2) under the above conditions. We suppose that (2) has at least one solution. Then*

$$\|y^* - x^*\|_\infty \le \frac{1}{1 - (L_1 + L_2)(b - a)} \left(\eta_1(b - a) + \eta_2\right),$$

for all solutions y^* of the equation (2).

Indeed,

$$\|y^* - x^*\|_\infty \le \frac{1}{1 - (L_1 + L_2)(b - a)} \|y^* - A_{a,b}y^*\|_\infty.$$

Moreover

$$\|y^*(t) - A_{a,b}y^*(t)\| \le$$

$$\int_a^b \|\tilde{K}(t, s, y^*(s), y^*(h(s))) - K(t, s, y^*(s), y^*(h(s)))\|ds + \|g(t) - \tilde{g}(t)\| \le$$

$$\eta_1(b - a) + \eta_2,\ \text{for all } t \in [\alpha, \beta].$$

Thus

$$\|y^* - A_{a,b}y^*\|_\infty \le \eta_1(b - a) + \eta_2,$$

and the above conclusion follows.

4 Ulam stability property

An important stability concept is that of Ulam-Hyers stability of the fixed point equation. The following notion was given by I.A. Rus in [14]. See also [16].

Definition 2 *Let (X, d) be a metric space and $A : X \to X$ be an operator. Then, the fixed point equation*

$$x = A(x) \tag{5}$$

is said to be Ulam-Hyers stable if there exists $c > 0$ such that, for any $\varepsilon > 0$ and any ε-solution $y^* \in X$ of (5), i.e.,

$$d\left(y^*, f\left(y^*\right)\right) \leq \varepsilon, \tag{6}$$

there exists a solution x^* of (5) such that

$$d\left(x^*, y^*\right) \leq c\varepsilon. \tag{7}$$

We have the following abstract result (see also I. A. Rus [14]) concerning the Ulam-Hyers stability of the fixed point equation (5).

Theorem 4 (Ulam-Hyers stability) *Let (X, d) be a metric space and $A : X \to X$ be a c-Picard operator. Then, the fixed point equation (5) is Ulam-Hyers stable.*

In what follows, we will consider the Ulam-Hyers stability property for equation (1).

Theorem 5 *Consider the equation (1). Suppose that all the assumptions of Theorem 2 hold. Then, the equation (1) is Ulam-Hyers stable.*

Proof. The conclusion follows by Theorem 2, Theorem 4 and the assertion 1) from Remark 1. □

5 Gronwall type lemmas

The following result is an abstract Gronwall lemma, see [13]. See also [11], [18], [7], [2], $\cdots$

Theorem 6 *Let $(X, d, \preceq)$ be an ordered metric space and $A : X \to X$ be an operator. We suppose:*

(i) A is increasing with respect to $\preceq$;

(ii) A is a Picard operator with $F_A = \{x^*\}$.

Then, the following conclusions hold:

(a) $x \in X,\ x \preceq A(x)$ implies $x \preceq x^*$;

(b) $x \in X,\ x \succeq A(x)$ implies $x \succeq x^*$.

Using the above abstract Gronwall lemma we can prove the following result.

Theorem 7 *Let us consider the equation (1) and suppose that all the hypotheses of Theorem 2 are satisfied. Let x^* be the unique solution of equation (1). In addition, we suppose:*

(i) $(\mathbb{B}, \|\cdot\|, \preceq)$ is an ordered Banach space;

(ii) the operator $K(t, s, \cdot, \cdot) : \mathbb{B}^2 \to \mathbb{B}^2$ is increasing, for each $(t, s) \in [\alpha, \beta]^2$.

Then, the following conclusions hold:

(1) if $x \in C([\alpha, \beta], \mathbb{B})$ is a solution of the inequality

$$x(t) \preceq \int_a^b K(t, s, x(s), x(h(s))ds + g(t),\ t \in [\alpha, \beta], \tag{8}$$

then $x \preceq x^*$.

(2) if $x \in C([\alpha, \beta], \mathbb{B})$ is a solution of the inequality

$$x(t) \succeq \int_a^b K(t, s, x(s), x(h(s))ds + g(t),\ t \in [\alpha, \beta], \tag{9}$$

then $x \succeq x^*$.

Proof. By our assumption (ii), it follows that the operator

$$A : C([\alpha, \beta], \mathbb{B}) \to C([\alpha, \beta], \mathbb{B}),$$

given by

$$Ax(t) := \int_a^b K(t, s, x(s), x(h(s))ds + g(t),\ t \in [\alpha, \beta]$$

is increasing with respect to the following partial ordering on $C([\alpha, \beta], \mathbb{B})$

$$x \leq y \text{ if and only if } x(t) \preceq y(t).$$

Moreover, by the proof of Theorem 2, we get that A is a Picard operator. The conclusion follows by the above theorem. $\square$

6 Data dependence with respect to a and b

In order to study the data dependence problem for equation with respect to a and b, we consider the following equation

$$x(t,a,b) = \int_a^b K(t,s,x(s,a,b),x(h(s),a,b)ds + g(t),\ t \in [\alpha,\beta], \tag{10}$$

where $\in [\alpha,\beta]$ and $(a,b) \in T := \{(u,v) : \alpha \le u \le v \le \beta\}$.

We also suppose that $K \in C([\alpha,\beta]^2 \times \mathbb{B}^2, \mathbb{B}), g \in C([\alpha,\beta],\mathbb{B})$ and $h \in C([\alpha,\beta],[\alpha,\beta])$.

We are looking for a solution $x^* \in C([\alpha,\beta] \times T, \mathbb{B})$. We endow the space $C([\alpha,\beta] \times T, \mathbb{B})$ with the norm $\|x\|_\infty := \max\limits_{[\alpha,\beta]\times T} \|x(t,u,v)\|$.

We notice that (10) is a fixed point equation with respect to the operator $A : C([\alpha,\beta] \times T, \mathbb{B}) \to C([\alpha,\beta] \times T, \mathbb{B})$ given by

$$Ax(t,a,b) := \int_a^b K(t,s,x(s,a,b),x(h(s),a,b))ds + g(t),\ t \in [\alpha,\beta].$$

We also suppose that the assumption (i) and (ii) in Theorem 2 holds, with the following modification of (ii): we suppose

$$(ii)'\ (L_1 + L_2)(\beta - \alpha) < 1.$$

It is obvious that $(ii)'$ implies (ii). Then, by Theorem 2 we obtain that A is a contraction. Thus, by Banach's contraction principle, we obtain the following result.

Theorem 8 *Let us consider the equation (10) under the above assumptions on K, g, h. We, additionally, suppose:*

(i) there exist $L_1, L_2 > 0$ such that

$$\|K(t,s,,u_1,v_1) - K(t,s,u_2,v_2)\| \le L_1\|u_1 - v_1\| + L_2\|u_2 - v_2\|,$$

for each $t,s \in [\alpha,\beta], u_1,v_1,u_2,v_2 \in \mathbb{B}$.

(ii)' $(L_1 + L_2)(\beta - \alpha) < 1$.

Then, we have the following conclusions:

(a) the equation (10) has in $C([\alpha,\beta] \times T, \mathbb{B})$ a unique solution x^*;

(b) the unique solution of the equation (10) is continuous with respect to a and b, where $(a,b) \in T$.

Proof. (a) the conclusion follows by Banach's contraction principle applied for A.

(b) If (ii)' takes place, then equation (1) has a unique solution for each $(a, b) \in T$. On the other hand, if x^* is the unique solution of the equation (10), then $x^*(\cdot; a, b) \in C([\alpha, \beta], \mathbb{B})$ is a solution of the equation (1). Thus, $x^*(\cdot; a, b)$ is continuous with respect to a and b, $(a, b) \in T$. $\square$

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