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REMARKS ON LYAPUNOV FUNCTIONS TO CAPUTO FRACTIONAL NEURAL NETWORKS *

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Dedicated to Professor Mihail Megan
on the occasion of his 70th anniversary

Abstract

One of the main properties of solutions of nonlinear Caputo fractional neural networks is stability and usually the direct Lyapunov method is used to study stability properties (usually these Lyapunov functions do not depend on the time variable). In this paper we give a brief overview of the most popular fractional order derivatives of Lyapunov functions and these derivatives are applied to various types of neural networks to illustrate their advantages/disadvantages. We show the quadratic Lyapunov functions and Lyapunov functions which do not depend directly on the time variable and their Caputo fractional derivatives are not applicable in some cases when one studies stability properties. Some suficient conditions using time dependent Lyapunov functions are obtained and illustrated on some particular nonlinear Caputo fractional neural networks.

MSC: 34A34, 34A08, 34D20

keywords: nonlinear Caputo fractional neural networks, Lyapunov functions, stability, fractional derivative of Lyapunov functions.

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1 Introduction

Fractional calculus arises naturally in physics, biological, chemical and engineering (see, for example, [8], [11], [25]) and fractional order models are used when one considers memory and hereditary properties of various materials and processes ([7]). Neural networks in biology, coupled lasers, wireless communication and power-grid networks in physics and engineering ([26], [22], [28]) are modeled by fractional order differential equations.

In controlling nonlinear systems, the Lyapunov second method provides a way to analyze the stability of the system without explicitly solving the differential equations and stability results concerning integer-order neural networks can be found in [12], [18], [29]. However Lyapunov stability theory for fractional order systems has not been developed until recently (see [15], [16]) so establishing stability sufficient criteria for fractional-order neural networks (FONN) is necessary and challenging. Fractional order Lyapunov stability theory was studied for various types of fractional neural networks using quadratic Lyapunov functions (see [7], [19], [20], [31]) and the uniform stability of fractional-order neural networks with delay was studied in [7]. Usually quadratic Lyapunov functions are used and it leads to a restriction to Lipschitz activation functions. In this paper we extend the fractional Lyapunov method to investigate stability behaviour of equilibrium points of neural networks. In particular with non-Lipschitz activations functions we apply Lyapunov functions depending directly on time. We show in the case of variable coefficients and non-Lipschitz activation functions in FONN that Lyapunov functions depending directly on the time variable and its Caputo fractional Dini derivative could be successfully applied to study stability.

In this paper we consider fractional derivatives with order $q \in (0,1)$. The Riemann–Liouville (RL) fractional derivative of order $q \in (0,1)$ of $m(t)$ is given by (see, for example, [23])

$$ {}_{t_0}^{RL}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} m(s) ds, \ t \geq t_0 $$

where $\Gamma(.)$ denotes the Gamma function.

The Caputo fractional derivative of order $q \in (0,1)$ is defined by (see, for example, [23])

$$ {}_{t_0}^{C}D_t^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m'(s) ds, \ t \geq t_0. \tag{1} $$

In this paper we present various definitions of fractional order derivatives of Lyapunov functions and compare them and using examples we discuss their advantages and disadvantages. Then fractional order Lyapunov stability theory is proposed to FONN. Some stability sufficient criteria for various types of fractional-order neural networks (using the appropriate fractional derivative of Lyapunov functions) are provided and illustrated with examples.

2 Lyapunov functions and their derivatives among the fractional differential equation

We first consider the derivative of Lyapunov functions among the studied fractional differential equation.

Consider the initial value problem (IVP) for the nonlinear Caputo fractional differential equations (FrDE)

$$ {}_{t_0}^{C}D_t^q x(t) = f(t, x(t)) \quad \text{for } t \in [t_0, t_0 + T), \qquad x(t_0) = x_0, \tag{2} $$

where $x \in \mathbb{R}^n$, $f \in C[[t_0, t_0 + T) \times \mathbb{R}^n, \mathbb{R}^n]$, $t_0 \in \mathbb{R}_+, x_0 \in \mathbb{R}^n$ is given initial data, $T \leq \infty$.

Let $x(t), t \in [t_0, t_0 + T)$, be a solution of the IVP for the FrDE (2) and let $V(t, x)$ be a Lyapunov function, i.e. $V(t, x) : [t_0, t_0 + T) \times \Delta \to \mathbb{R}_+$ is continuous on $[t_0, t_0 + T) \times \Delta$ and it is locally Lipschitzian with respect to its second argument, where $\Delta \subset \mathbb{R}^n, \ \ 0 \in \Delta$.

In the literature there are three types of derivatives of Lyapunov functions among solutions of fractional differential equations used to study stability properties:

- *first type*– Let $x(t)$ be a solution of IVP for FrDE (2). Then the **Caputo fractional derivative** of the Lyapunov function $V(t, x(t))$ among the FrDE (2) is defined by

$$ {}_{t_0}^{C}D_t^q V(t, x(t)) = \frac{1}{\Gamma(1-q)} \int\limits_{t_0}^{t} (t-s)^{-q} \frac{d}{ds}\Big(V(s, x(s))\Big) ds, $$
$$ t \in (t_0, t_0 + T). \tag{3} $$

  This type of derivative is applicable for continuously differentiable Lyapunov functions. It is used mainly for quadratic Lyapunov functions to study several stability properties of fractional differential equations (see, for example, [15]).

- *second type*– this type of derivative of $V(t,x)$ among FrDE (2) is introduced in [13]:

$$D^{+}_{(2)}V(t,x) = \limsup_{h\to 0} \frac{1}{h^q}\Big[V(t,x) - V(t-h, x-h^q f(t,x))\Big],$$
$$t \in (0,T),\ x \in \Delta. \tag{4}$$

Note the operator defined by (4) has no memory (the memory is typical for fractional derivatives).

Remark 1. *In general $D^{+}_{(2)}V(t,x(t)) \neq {}^{C}_{t_0}D^q_t V(t,x(t))$ where $x(t)$ is a solution of (2).*

Now, let us recall the remark in [9] concerning definition (4) where $V(t-h, x-h^q f(t,x))$ is defined by

$$V(t-h, x-h^q f(t,x)) = \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\ {}_qC_r V(t-rh, x-h^q f(t,x)),$$

where ${}_qC_r = \frac{q(q-1)\ldots(q-r+1)}{r!}$.

Following this notation the fractional derivative of the Lyapunov function among the FrDE (2) is defined by

$$\mathcal{D}^{+}_{(2)}V(t,x) = \limsup_{h\to 0} \frac{1}{h^q}\Big[V(t,x) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\ {}_qC_r V(t-rh, x-h^q f(t,x))\Big]. \tag{5}$$

The derivative (5) has memory and it depends on the initial time t_0. It is closer to both, the Grunwald-Letnikov fractional derivative and the Riemann-Liouville fractional derivative, than to the Caputo fractional derivative of a function. It does not depend on the initial value $V(t_0, x_0)$ which is typical for the Caputo derivative. We will call the derivative (5) the **Dini fractional derivative** of the Lyapunov function. The Dini fractional derivative is applicable for continuous Lyapunov functions.

Remark 2. *In the general case $\mathcal{D}^{+}_{(2)}V(t,x) \neq D^{+}_{(2)}V(t,x)$ (see Example 1, Case 2.1 and Case 2.2).*

- *third type* - the derivative of the Lyapunov function $V(t,x)$ is defined by:

$$
\begin{aligned}
{}^{c}_{(2)}D^{q}_{+}V(t,x;t_0,x_0) = \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{ & V(t,x) - V(t_0,x_0) \\
- \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}\, {}_{q}C_r \Big(& V(t-rh, x-h^q f(t,x)) - V(t_0,x_0)\Big)\Big\}, \qquad (6)
\end{aligned}
$$

for $t\in(t_0,t_0+T)$

or its equivalent

$$
\begin{aligned}
{}^{c}_{(2)}D^{q}_{+}V(t,x;t_0,x_0) &= \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{ V(t,x) \\
&+ \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r}\, {}_{q}C_r V(t-rh, x-h^q f(t,x))\Big\} \qquad (7) \\
&- \frac{V(t_0,x_0)}{(t-t_0)^q\Gamma(1-q)} \quad \text{for } t\in(t_0,t_0+T).
\end{aligned}
$$

The derivative (7) depends significantly on both the fractional order q and the initial data (t_0,x_0) of IVP for FrDE (2) and this type of derivative is close to the idea of the Caputo fractional derivative of a function.

We call the derivative given by (6) or its equivalent (7) the **Caputo fractional Dini derivative**. This type of derivative is applicable for continuous Lyapunov functions.

Remark 3. *The equality* ${}^{c}_{(2)}D^{q}_{+}V(t,x;t_0,x_0) = \mathcal{D}^{+}_{(2)}V(t,x) - \frac{V(t_0,x_0)}{(t-t_0)^q\Gamma(1-q)}$, $t\in(t_0,t_0+T)$, *holds and for any* $t\in(t_0,t_0+T)$, $x_0\in\mathbb{R}^n$

$$
{}^{c}_{(2)}D^{q}_{+}V(t,x;t_0,x_0) = \mathcal{D}^{+}_{(2)}V(t,x), \quad \textit{if } V(t_0,x_0)=0
$$

$$
{}^{c}_{(2)}D^{q}_{+}V(t,x;t_0,x_0) < \mathcal{D}^{+}_{(2)}V(t,x), \quad \textit{if } V(t_0,x_0)>0.
$$

From the literature we note that one of the sufficient conditions for stability is connected with the sign of the derivative of the Lyapunov function. **EXAMPLE 1**. Consider the IVP for the scalar linear FrDE

$$
{}^{C}_{0}D^{q}_{t}x(t) = g(t)x \text{ for } t>0, \quad x(0)=x_0, \qquad (8)
$$

where $q \in (0,1)$, $g(t) = -0.5\frac{{}_0^{RL}D^q\left(\sin^2(t)+0.1\right)}{sin^2(t)+0.1}$ (the graph of the function $g(t)$ for various values of q is given on Figure 1). The sign of the function $g(t)$ is changeable.

Case 1. Consider the quadratic Lyapunov function (used in stability of neural networks [7], [19], [20], [31]) i.e. $V(t,x) = x^2$. Let $x(t)$ be a solution of IVP for FrDE (8). Then according to [10] we get

$$\begin{aligned}{}_0^C D_t^q V(t,x(t)) &= \frac{2}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\, x(s)x'(s)ds \le 2x(t)\; {}_0^C D_t^q x(t)\\ &= 2(x(t))^2 g(t).\end{aligned}$$

The sign of ${}_0^C D_t^q V(t,x(t))$ is changeable.

Case 2. Consider the function $V(t,x) = (\sin^2(t)+0.1)x^2$.

Case 2.1: Caputo fractional derivative. Let $x(t)$ be a solution of IVP for FrDE (8). From (3) we obtain

$${}_0^C D_t^q V(t,x(t)) = \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}(2(\sin^2(s)+0.1)x(s)x'(s)+\sin(2s)x^2(s))ds.$$

The fractional derivative of this function V among IVP for FrDE (8) is difficult to obtain so it is difficult to discuss its sign.

Case 2.2: Use formula (4) and obtain

$$\begin{aligned}D^+_{(8)}V(t,x) &= \limsup_{h\to 0}\frac{1}{h^q}\Big[(\sin^2(t)+0.1)x^2 - (\sin^2(t-h)+0.1)(x-h^q x g(t))^2\Big]\\ &= 2x^2(sin^2(t)+0.1)g(t),\end{aligned}$$

i.e. the sign of the derivative $D^+_{(8)}V(t,x)$ is changeable.

Case 2.3: Dini fractional derivative. From formula (5) we obtain

$$\begin{aligned}
\mathcal{D}^{+}_{(8)}V(t,x) &= \limsup_{h\to 0} \frac{1}{h^q}\Big[(\sin^2(t)+0.1)x^2 \\
&- \sum_{r=1}^{[\frac{t}{h}]}(-1)^{r+1}\ {}_qC_r(\sin^2(t-rh)+0.1)(x-h^q x g(t))^2\Big] \\
&= x^2 \limsup_{h\to 0} \frac{1}{h^q}\Big[(\sin^2(t)+0.1) \\
&+ \sum_{r=1}^{[\frac{t}{h}]}(-1)^{r}\ {}_qC_r(\sin^2(t-rh)+0.1)(1-h^q g(t))^2\Big] \\
&= x^2 \limsup_{h\to 0} \frac{1}{h^q}\Big[(\sin^2(t)+0.1)(1-(1-h^q g(t))^2) \\
&+ (1-h^q g(t))^2 \sum_{r=0}^{[\frac{t}{h}]}(-1)^{r}\ {}_qC_r(\sin^2(t-rh)+0.1)\Big] \\
&= 2x^2 g(t)(\sin^2(t)+0.1) + x^2\ {}_0^{RL}D^q\Big(\sin^2(t)+0.1\Big) = 0.
\end{aligned}$$

Case 2.4: Caputo fractional Dini derivative. According to Remark 3 and Case 2.3 the inequality

$${}^{c}_{(8)}D^{q}_{+}V(t,x;0,x_0) = \mathcal{D}^{+}_{(8)}V(t,x) - \frac{0.1x_0^2}{t^q\Gamma(1-q)} = -\frac{0.1x_0^2}{t^q\Gamma(1-q)} \leq 0$$

holds.

Therefore, for (8) both the Dini fractional derivative and the Caputo fractional Dini derivative seem to be more applicable than the Caputo fractional derivative of Lyapunov function.

□

Remark 4. *The above example notes that the quadratic function for studying stability properties of neural network might not be successful (especially when the right side parts depend directly on the time variable). Formula (4) is not appropriate for applications to fractional equations. The most general derivatives for non-homogenous fractional differential equations are Dini fractional derivatives and Caputo fractional Dini derivatives.*

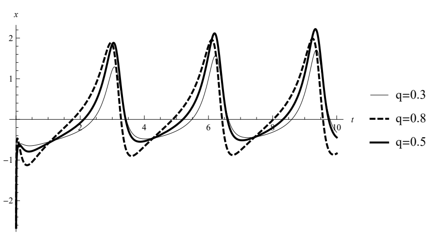

Figure 1. Example 1. Graph of the function $g(t)$ for various values of q.

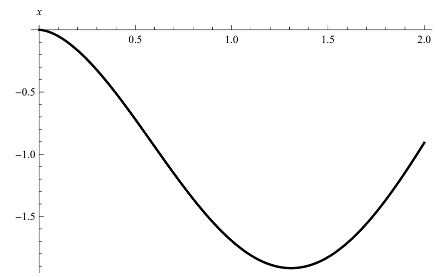

Figure 2. Example 2. Graph of the fractional derivative ${}_{0}^{C}D_t^q\Big(-(sin(t))^2\Big)$.

3 Stability for fractional-order neural networks

3.1 System Description

Consider the general model of FONN

$$ {}_{0}^{C}D_t^q x_i(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j(x_j(t)) + I_i(t) \quad \text{for } t > 0, \quad i = 1, 2, \dots n, \tag{9} $$

or equivalently

$$ {}_{0}^{C}D_t^q x(t) = -C(t)x(t) + A(t)f(x(t)) + I(t) \quad \text{for } t > 0 \tag{10} $$

where n represents the number of units in the network

$$ x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n, \quad C(t) = diag(c_i(t)), \quad A(t) = \{a_{ij}(t)\} $$

corresponds to the connection of the i-th neuron to the j-th neuron at time t, $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \dots, f_n(x_n(t))]^T$ is the activation function of the neurons, and $I = [I_1, I_2, \dots, I_n]^T$ is an external bias vector.

Remark 5. *The stability analysis of FONN (9) is studied in [20] in the special case* $c_j(t) \equiv c_j = const,\ j = 1, 2, \dots, n$ *but the main result, Lemma 5 (in [20]), is not correct. For example, for the function* $y(t) = -(sin(t))^2$ *the fractional derivative* ${}_{0}^{C}D_t^q y(t) \leq 0,\ t \in [0, 2]$ *(see Figure 2) but the function* $y(t)$ *is not decreasing on* $[0, 2]$.

Definition 1. *A vector* $x^* \in \mathbb{R}^n$ *is an equilibrium point of Caputo FONN (10), if and only if the equality* $0 = -Cx^* + A(t)f(x^*) + I(t)$ *holds for all* $t > 0$.

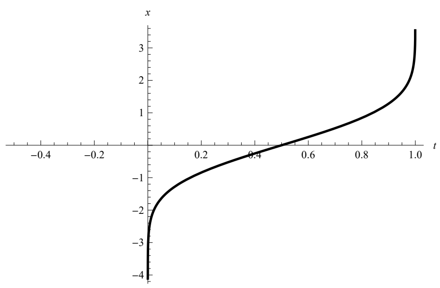

Figure 3. Example 2. Graph of the function $f(t) = \sqrt{2}\ \mathrm{erf}^{-1}(2t-1)$.

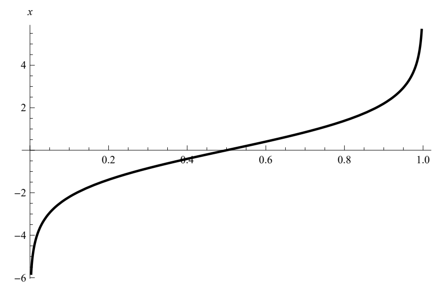

Figure 4. Example 2. Graph of the function $f(u) = \log\left(\frac{u}{1-u}\right)$.

We will discuss the equilibrium points on some FONN with various activation functions. It will be useful for stability analysis.

EXAMPLE 2. Let $n = 1$ and consider the scalar linear FONN

$$ {}_0^C D_t^q x(t) = -cx(t) + a(t)f(x(t)) + I \text{ for } t > 0, \tag{11}$$

where $q \in (0,1)$, c is a constant.

Case 1. Let $I = 0.5\pi c$ and the activation function be the cosine function $f(u) = \cos(u)$ (see [24]).

The point $x^* = 0.5\pi$ is an equilibrium point of FONN (11) because $-c0.5\pi + a(t)f(0.5\pi) + 0.5\pi c = 0$ for all $t > 0$.

Case 2. Let $I = 0.5c$ and the activation function be the Probit function $f(u) = \sqrt{2}\ \mathrm{erf}^{-1}(2u-1)$ (see Figure 3) where $\mathrm{erf}(u) = \frac{2}{\sqrt{\pi}}\int_0^u e^{-t^2}dt$ is the error function. Then $\mathrm{erf}^{-1}(0) = 0$.

The point $x^* = 0.5$ is an equilibrium point of FONN (11) because $-c0.5 + a(t)f(0.5) + 0.5c = 0$ for all $t > 0$.

Case 3. Let $I = 0.5c$ and the activation function be the Logit function $f(u) = \log\left(\frac{u}{1-u}\right)$ (see Figure 4). Then $f(0.5) = 0$.

The point $x^* = 0.5$ is an equilibrium point of Caputo FONN (11) because $-c0.5 + a(t)f(0.5) + 0.5c = 0$ for all $t > 0$.

□

EXAMPLE 3. Let $n = 2$ and consider the system of FONN

$$\begin{aligned} {}_0^C D_t^q x_1(t) &= -c_1 x_1(t) + a_{11}(t)\arctan(x_1(t)) + a_{12}\cos(x_2(t)) + I_1, \\ {}_0^C D_t^q x_2(t) &= -c_2 x_2(t) + a_{21}(t)\arctan(x_1(t)) + a_{22}\cos(x_2(t)) + I_2 \text{ for } t > 0, \end{aligned} \tag{12}$$

where $q \in (0,1)$, $c_i, a_{i,j}, i,j = 1,2$ are constants.

The point $x^* = (0,0)$ is an equilibrium point of Caputo FONN (12) if $I_1 = -a_{12}, I_2 = -a_{22}$.

The FONN (12) is considered in the special case $q = 0.88, a_{11}(t) \equiv 0.5, a_{21}(t) \equiv -0.9, a_{12} = 1, a_{22} = -0.7$ in [19].

$\square$

Assumption A1. Let the Caputo FONN (9) have an equilibrium point x^*.

If assumption A1 is satisfied then we can shift the equilibrium point x^* of system (9) to the origin. The transformation $y(t) = x(t) - x^*$ is used to put system (9) in the following form:

$$
\begin{aligned}
&{}_0^C D_t^q y_i(t) = -c_i(t) y_i(t) + \sum_{j=1}^{n} a_{ij}(t)(f_j(y_j(t) + x_j^*) - f_j(x_j^*)) \\
&\text{for } t > 0, \quad i = 1, 2, \dots n,
\end{aligned} \tag{13}
$$

or equivalently

$$
{}_0^C D_t^q y(t) = -C(t) y(t) + A(t) F(y(t)) \quad \text{for } t > 0 \tag{14}
$$

where $F(u) = [F_1(u_1), F_2(u_2), \dots, F_n(u_n)]^T$,
$F_j(u_j) = f_j(u_j + x_j^*) - f_j(x_j^*), j = 1, 2, \dots, n.$

3.2 Some results for Caputo fractional differential equations.

We will give some results for Caputo fractional derivatives and Caputo fractional differential equations which will be used for our main results concerning stability.

Lemma 1. *([10]). Let $P \in \mathbb{R}^{n\times n}$ be constant, symmetric and positive definite matrix and $X(t) : \mathbb{R}_+ \to \mathbb{R}^n$ be a function with the Caputo fractional derivative existing.*

Then $\frac{1}{2}\, {}_0^C D_t^q \Big(x^T(t) P x(t) \Big) \leq x^T(t) P\, {}_0^C D_t^q x(t), \quad t \geq 0.$

Lemma 2. *(Theorem 11 [16]). Let $x = 0$ be an equilibrium point for (2) with $T = \infty$. Assume there exists a Lyapunov function $V(t, x)$ such that*

(i) $\alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||), \quad t \geq 0, x \in \Delta,$

(ii) ${}_0^C D_t^\beta V(t, x(t)) \leq -\alpha_3(||x(t)||), \quad t > 0$ *where $\beta \in (0, 1)$, $x(t)$ is a solution of (2) and functions $\alpha_i \in C([0,\infty), [0,\infty))$, $i = 1, 2, 3$ are strictly increasing and $\alpha_i(0) = 0$.*

Then the equilibrium point of (2) is asymptotically stable.

For the Caputo fractional Dini derivative we have:

Lemma 3. *(Theorem 3[3]).Assume $x = 0$ be an equilibrium point for (2) with $T = \infty$. Assume there exists a Lyapunov function $V(t, x)$: $V(t, 0) = 0,\ t \geq 0$, such that*

(i) $\alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||), \quad t \geq 0, x \in \Delta,$

(ii) ${}^{c}_{(2)}D^q_+ V(t, x; 0, x_0) \leq -\alpha_3(||x||), \quad t > 0,\ x, x_0 \in \Delta$ *where functions* $\alpha_i \in C([0, \infty), [0, \infty)),\ i = 1, 2, 3$ *are strictly increasing and* $\alpha_i(0) = 0$.

Then the equilibrium point of (2) is asymptotically stable.

3.3 Stability Analysis.

We will study stability properties of several different types of FONN (2) using different types of Lyapunov functions and their fractional derivatives given in Section 2.

3.3.1. Lipschitz activation functions and quadratic Lyapunov functions.

We assume the following:

Assumption A2. Let the activation function of the neurons be Lipschitz, i.e. there exist positive numbers $L_i > 0,\ i = 1, 2, \dots, n$ such that $|f_i(u) - f_i(v)| \leq L_i|u - v|$, for $u, v \in \mathbb{R}$.

Assumption A3. There exists positive numbers $M_{i,j}$ such that $|a_{i,j}(t)| \leq M_{i,j}$, for $t > 0$.

Assumption A4. The inequality $\lambda > 0.5L_{max}$ holds, where $\alpha_i = \sum_{j=1}^{n} \big(M_{ji}L_i + M_{ij}L_j\big)$, $L_{max} = \max\{\alpha_i,\ i = 1, 2, \dots, n\}$, $\lambda = \min\{c_j,\ j = 1, 2, \dots, n\}$.

Remark 6. *If assumption A2 is satisfied then the function F in FONN (14) satisfies $|F_j(u)| \leq L_j|u|,\ j = 1, 2, \dots, n$ for any $u \in \mathbb{R}$.*

The case, when the functions of the connection of the i-th neuron to the j-th neuron at time t in FONN (9) are bounded is studied using the quadratic Lyapunov function (for integer order see [21]). We will provide only one result.

Theorem 1. *Let $c_j(t) \equiv c_j > 0,\ j = 1, 2, \dots, n$ and assumptions A1, A2, A3, A4 are satisfied.*

Then the equilibrium point x^* of Caputo FONN (9) is asymptotically stable.

P r o o f: Consider the quadratic functions $V(t, x) = x^T x$. Let $x(t)$ be a solution of the FONN (14). Then applying Lemma 1 we get

$$
\begin{aligned}
{}_0^C D_t^q V(t, x(t)) &\le -2 \sum_{i=1}^n c_i x_i^2(t) + 2 \sum_{i=1}^n \sum_{j=1}^n |a_{ij}||F_j(x_j(t))|\ |x_i(t)| \\
&\le -2\lambda \sum_{i=1}^n x_i^2(t) + \sum_{i=1}^n \sum_{j=1}^n M_{ij} L_j (x_j^2(t) + x_i^2(t)) \\
&\le -2\lambda \sum_{i=1}^n x_i^2(t) + \sum_{i=1}^n x_i^2(t) \Big(\sum_{j=1}^n \big(M_{ji} L_i + M_{ij} L_j\big) \Big) \\
&\le -2 \Big(\lambda - 0.5 L_{max} \Big) \sum_{i=1}^n x_i^2(t).
\end{aligned}
\tag{15}
$$

Inequality (15) and Lemma 2 prove the claim.

□

EXAMPLE 4. Consider the system of FONN (12). The point $x^* = (0, 0)$ is an equilibrium point of Caputo FONN (12) if $I_1 = -a_{12}, I_2 = -a_{22}$ (see Example 3).

In the special case $q = 0.88, a_{11}(t) \equiv 0.5, a_{21}(t) \equiv -0.9, a_{12} = 1, a_{22} = -0.7$ in [19] the authors prove the solution converges to zero asymptotically for $c_1 = 2, c_2 = 3$.

Now, we will prove that for any $q \in (0, 1)$ and $c_i > 1.65, i = 1, 2$ the zero equilibrium is asymptotically stable. Indeed, the functions $\arctan(x)$ and $\cos(x)$ are Lipschitz with $L = 1$, $\alpha_1 = 2.9, \alpha_2 = 3.3$, $L_{max} = 3.3$ and for $c_i > 1.65$ and according to Theorem 1 the claim is true.

□

3.3.2. Non-Lipschitz activation functions and quadratic Lyapunov functions.

There are many types of activation functions which are not Lipschitz (see Example 2, Cases 2 and 3). In this case we assume:

Assumption A5. There exists a function $\xi \in C(\mathbb{R}_+, \mathbb{R})$ such that for any solution $x(t)$ of the FONN (14) and $i = 1, 2, \dots, n$ the inequality $x_i(t) \sum_{j=1}^n a_{i,j}(t)(f_j(x_j(t) + x_j^*) - f_j(x_j^*) \le \xi(t) x_i^2(t)$ holds.

Assumption A6. There exists a function $\eta \in C(\mathbb{R}_+, (0, \infty))$ such that $c_i(t) \geq \eta(t),\ i = 1, 2, \ldots, n,\ t \geq 0.$

Theorem 2. *Let assumptions A1, A5, A6 be satisfied with* $\xi(t) \leq \eta(t),\ t \geq 0$.

Then the equilibrium point x^* *of Caputo FONN (9) is asymptotically stable.*

P r o o f: Consider the quadratic functions $V(t, x) = x^T x$. Let $x(t)$ be a solution of the FONN (14). Then applying Lemma 1 we get for the Caputo fractional derivative

$$\begin{aligned}
{}_0^C D_t^q V(t, x(t)) &\leq -2 \sum_{i=1}^{n} c_i(t) x_i^2(t) + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}(t) x_i(t) F_j(x_j(t)) \\
&= -2 \sum_{i=1}^{n} \Big(c_i(t) x_i^2(t) - x_i(t) \sum_{j=1}^{n} a_{i,j}(t) F_j(x_j(t)) \Big) \qquad (16) \\
&\leq -2 \Big(\eta(t) - \xi(t) \Big) \sum_{i=1}^{n} x_i^2(t).
\end{aligned}$$

From inequality (16) and Lemma 2 the claim follows.

□

EXAMPLE 5. Let $n = 1$ and consider the scalar nonlinear FONN

$${}_0^C D_t^q x(t) = -c x(t) + a(t) f(x(t)) + I \text{ for } t > 0, \qquad (17)$$

where $q \in (0, 1)$, $I = 0.5c$, $c > 0$, $a(t) \in C([0, \infty), (\infty, 0])$ and the activation function $f(t)$ is the Probit function or the Logit function (see Example 2). Then the equation (17) has an equilibrium point $x^* = 0.5$ (see Example 2). Both activation functions are not Lipschitz and Theorem 1 cannot be applied. Then using $x f(x(t) + 0.5) \geq 0$ condition A5 is reduced to $a(t) x(t) f(x(t) + 0.5) \leq 0$ for any solution of FONN (17), and therefore assumption A5 is satisfied with $\xi(t) \equiv 0 < c$. Then according to Theorem 2 the equilibrium point $x^* = 0.5$ of FONN (17) is asymptotically stable for all $c > 0$.

□

3.3.3. Non-Lipschitz activation functions and time-depended Lyapunov functions.

In the case the function $\eta(t)$ in Assumption A6 is not enough large so we introduce

Assumption A7. There exists a continuous positive function $g(t) \in C([0,\infty),\mathbb{R}_+)$ such that $0 < \alpha \leq g(t) \leq \beta$ and the fractional derivative ${}_0^{RL}D_t^q g(t)$ exists for $t > 0$.

In this case Assumption A5 could be weakened to

Assumption A8. There exists a function $\xi \in C(\mathbb{R}_+,\mathbb{R})$ such that for any point $x \in \mathbb{R}^n,\ x = (x_1, x_2, \dots, x_n)$, and $i = 1, 2, \dots, n$ the inequality $x_i \sum_{j=1}^n a_{i,j}(t)(f_j(x_j + x_j^*) - f_j(x_j^*) \leq \xi(t)x_i^2$ holds.

Theorem 3. *Let the assumptions A1, A6, A7, A8 be satisfied and there exists a constant $K > 0$ such that*

$$-g(t)\eta(t) + g(t)\xi(t) + 0.5\ {}_0^{RL}D_t^q g(t) \leq -K. \tag{18}$$

Then the equilibrium point x^* of Caputo FONN (9) is asymptotically stable.

P r o o f: In this case the quadratic function $V(t,x) = \sum_{j=1}^n x_i^2,\ x = (x_1, x_2, \dots, x_n)$ does not work.

Consider the function $V(t,x) = g(t)\sum_{i=1}^n x_i^2,\ x = (x_1, x_2, \dots, x_n)$ where the function $g(t)$ is defined in Assumption A7. Then according to Assumption A7 condition (i) of Lemma 3 is satisfied with $\alpha_1(u) = \alpha u$ and $\alpha_2(u) = \beta u$. Also, we get the following inequality for the fractional Dini derivative

$$\begin{aligned} \mathcal{D}^+_{(14)}V(t,x) &= -2\sum_{i=1}^n g(t)c_i(t)x_i^2 + 2\sum_{i=1}^n g(t)x_i \cdot \\ &\quad \cdot \sum_{j=1}^n a_{i,j}(t)F_j(x_j) + {}_0^{RL}D_t^q g(t)\sum_{i=1}^n x_i^2 \\ &\leq \Big(-2g(t)\eta(t) + 2g(t)\xi(t) + {}_0^{RL}D_t^q g(t)\Big)\sum_{j=1}^n x_j^2 \\ &\leq -2K\sum_{j=1}^n x_j^2. \end{aligned} \tag{19}$$

According to Remark 3 and inequality (19) we get the inequality

$${}_{(14)}^{c}D_+^q V(t,x;0,x_0) = \mathcal{D}^+_{(14)}V(t,x) - \frac{V(0,x_0)}{t^q\Gamma(1-q)} \leq -2K\sum_{j=1}^n x_j^2,$$

i.e. condition (ii) of Lemma 3 is satisfied. Applying Lemma 3 we prove the claim.

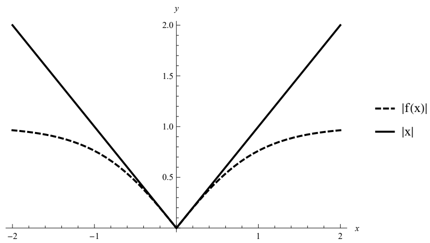

Figure 5. Example 6. Graph of the functions $|f(x)| = |\tanh(x)|$ and $|x|$.

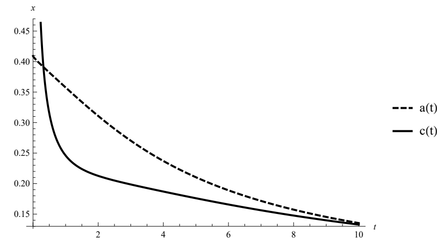

Figure 6. Example 6. Graph of the functions $a(t)$ and $c(t)$ for $q = 0.3$.

$\square$

EXAMPLE 6. Let $n = 1$ and consider the scalar linear FONN (11) with $c(t) = \frac{0.55}{t^q\Gamma(1-q)(E_q(-t^q)+0.1)} + \frac{0.005}{E_q(-t^q)+0.1}$, $I(t) = -c(t)$, $a(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$: $a(t) = \frac{0.25E_q(-t^q)}{E_q(-t^q)+0.1}$, the activation function is the Continuous Tan-Sigmoid Function $f(u) = \tanh(u) = \frac{e^u - e^{-u}}{e^u + e^{-u}}$ and the equilibrium point $x^* = 0$.

Theorem 1 is not applicable since the coefficient before x is not a constant.

Let $x(t)$ be a solution of FONN (11). Then using the inequality $|f(x)| \leq |x|$, $x \in \mathbb{R}$ (see Figure 5) we get $x(t)a(t)(f(x(t)) - f(0)) \leq a(t)x^2(t)$, i.e. Assumption A5 is satisfied with $\xi \equiv a(t)$. However the inequality $\xi(t) = a(t) = \frac{0.45E_q(-t^q)}{E_q(-t^q)+0.1} \leq \eta(t) = c(t) = \frac{0.55}{t^q\Gamma(1-q)(E_q(-t^q)+0.1)} + \frac{0.005}{E_q(-t^q)+0.1}$ is not satisfied (see Figure 6 for $q = 0.3$). Therefore, Theorem 2 cannot be applied.

Consider the function $g(t) = (E_q(-t^q) + 0.1)$. Then we get the inequality

$$\begin{aligned}
&-g(t)\eta(t) + g(t)\xi(t) + 0.5\, {}_0^{RL}D_t^q g(t) \\
&= -(E_q(-t^q) + 0.1)\Big(\frac{0.55}{t^q\Gamma(1-q)(E_q(-t^q)+0.1)} + \frac{0.005}{E_q(-t^q)+0.1}\Big) \\
&\quad + (E_q(-t^q) + 0.1)\frac{0.45E_q(-t^q)}{E_q(-t^q)+0.1} + 0.5\, {}_0^{RL}D_t^q\Big(E_q(-t^q) + 0.1\Big) \\
&= -\frac{0.55}{t^q\Gamma(1-q)} - 0.005 + 0.45E_q(-t^q) + 0.5\Big(- E_q(-t^q) + \frac{1.1}{t^q\Gamma(1-q)}\Big) \\
&= -\frac{0.55}{t^q\Gamma(1-q)} - 0.005 + 0.45E_q(-t^q) - 0.5E_q(-t^q) + \frac{0.55}{t^q\Gamma(1-q)} \\
&= -0.005 - 0.05E_q(-t^q) = -0.05\Big(E_q(-t^q) + 0.1\Big).
\end{aligned} \tag{20}$$

According to Theorem 3 the equilibrium point $x^* = 0$ of Caputo FONN (11) is asymptotically stable.

Therefore, in the case of variable coefficients c_i and non-Lipschitz activation function in the FONN the Lyapunov function depending directly on the time variable and its Caputo fractional Dini derivative could be successfully applied to study the stability.

□

Conclusions

In this paper, stability of neural networks with time-varying functions of connections and Caputo fractional derivative is studied. In controlling nonlinear systems, the Lyapunov second method provides a way to analyze the stability of the system without solving differential equations. Usually quadratic Lyapunov functions are used and it leads to a restriction to Lipschitz activation functions. In this paper we extend the fractional Lyapunov method to investigate stability behaviour of equilibrium points of neural networks. In particular with non-Lipschitz activations functions we apply Lyapunov functions depending directly on time. We show in the case of variable coefficients and non-Lipschitz activation functions in FONN that Lyapunov functions depending directly on the time variable and its Caputo fractional Dini derivative could be successfully applied to study stability.

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