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TIME FRACTIONAL OSEEN PROBLEM FOR VISCOUS FLUIDS *

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Dedicated to Professor Mihail Megan
on the occasion of his 70th anniversary

Abstract

Time fractional Oseen problem is analytically solved for viscous fluids. Exact solutions are obtained for the dimensionless velocity field and the corresponding non-trivial shear stress and circulation. These solutions, as it was to be expected, reduce to the non-dimensional forms of classical solutions when the fractional parameter tends to one. The decay of potential vortex and the diffusion of vorticity under the influence of fractional parameter are graphically underlined and discussed. The power of vortex as well as the diffusion of vorticity are stronger for fractional in comparison to ordinary fluids. In all cases the vortex decreases in time and space.

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1 Introduction

Decay of a potential vortex in viscous fluids (Oseen, 1911) has been elegantly solved by Zierep [1] using similarity by transformation of variables. He found the exact solution

$$\omega(r,t) = \frac{\Gamma_0}{2\pi r}\left[1 - exp\left(-\frac{r^2}{4\nu t}\right)\right], \tag{1}$$

corresponding to the problem

$$\frac{\partial \omega(r,t)}{\partial t} = \nu\left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t),\ \omega(r,0) = \frac{\Gamma_0}{2\pi r};\ r,t>0, \tag{2}$$

where $\omega(r,t)$ is the rotational component of velocity and ν is the kinematic viscosity of the fluid. The initial distribution of velocity, as it results from Eq. $(2)_2$, is that of a potential vortex of circulation Γ_o. Since the flow domain is unbounded, the natural conditions

$$\omega(r,t), \frac{\partial \omega(r,t)}{\partial r} \to 0\ as\ r\to\infty\ and\ t>0, \tag{3}$$

have been also used. They implies that the fluid is quiescent at infinity and there is no shear in the free streams [2,3]. The circulation $\Gamma(r,t)$ on a circle of radius r, which is of further interest, is given by

$$\Gamma(r,t) = 2\pi r\omega(r,t) = \Gamma_0\left[1 - exp\left(-\frac{r^2}{4\nu t}\right)\right]. \tag{4}$$

The results of Zierep [1] have been also extended to non-Newtonian fluids [4-6]. In [4] and [5], for instance, the temperature distribution and the nontrivial shear stress corresponding to second grade, respectively Maxwell fluids have been analytically determined. The non-trivial shear stress corresponding to this problem, as it results from [5, Eq. (4.5)], is given by

$$\tau(r,t) = \mu\left\{\frac{\partial \omega(r,t)}{\partial r} - \frac{1}{r}\omega(r,t)\right\} = \frac{\mu\Gamma_0}{\pi r^2}\left\{\left(1+\frac{r^2}{4\nu t}\right)exp\left(-\frac{r^2}{4\nu t}\right) - 1\right\}, \tag{5}$$

where μ is the fluid viscosity. Consequently, $r\omega(r,t)$ and $r^2\tau(r,t)$ like the circulation $\Gamma(r,t)$ depend of r and t only by means of the similarity variable $\frac{r}{\sqrt{2\nu t}}$. However, none of the previous works took into consideration the fractional calculus.

The purpose of this note is to extend the Oseen problem for viscous fluids to a time fractional model. More exactly, we want to use the advantages of one of the most modern and recent definitions of the non-integer order derivative in order to provide exact solutions for the time fractional Oseen problem. As a motivation, we remember the fact that the fractional models are more flexible in describing the complex behaviour of many materials and the first authors who applied fractional derivatives in elasticity or viscoelasticity are Germant [7], respectively Bagley and Torvik [8]. On the other hand, the memory formulism, which is connected to viscoelastic fluids, can be brought to light using fractional derivatives [9]. Furthermore, Makris et al. [10] used experimental data to calibrate a fractional derivative Maxwell model. More exactly, they determined the value of the fractional parameter so that the predicted material properties to be in excellent agreement with the experimental results. Interesting observations regarding the importance and the multiple applications of fractional derivatives can be found in the recent paper of Sheoran and Kundu [11].

2 Statement of the Problem

In order to develop solutions free of the geometry of flow, we use the next non-dimensional variables and functions:

$$t^* = \frac{t}{t_0},\ r^* = \frac{r}{\sqrt{\Gamma_0 t_0}},\ \omega^* = 2\pi\sqrt{\frac{t_0}{\Gamma_0}}\omega,\ \Gamma^* = \frac{\Gamma}{\Gamma_0},\ \nu^* = \frac{\nu}{\Gamma_0},\ \tau^* = \frac{t_0}{\mu}\tau, \quad (6)$$

where t_0 is a reference time. Substituting Eqs. (6) in Eqs. (2) and dropping out the star notation, we find that

$$\frac{\partial\omega(r,t)}{\partial t} = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t),\ \omega(r,0) = \frac{1}{r};\ r,t>0. \quad (7)$$

The natural conditions (3) maintain the same form while the dimensionless circulation and the shear stress are given by

$$\Gamma(r,t) = r\omega(r,t),\ \tau(r,t) = \frac{1}{2\pi}\left\{\frac{\partial\omega(r,t)}{\partial r} - \frac{1}{r}\omega(r,t)\right\}. \quad (8)$$

The corresponding fractional model is based on the non-integer order partial differential equation

$$^{CF}D_t^{\alpha}\omega(r,t) = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t);\ r,t>0, \quad (9)$$

with the conditions (3) and $(7)_2$. Here the Caputo-Fabrizio time fractional derivative [12]

$$^{CF}D_t^{\alpha}\omega(r,t) = \frac{1}{1-\alpha}\int_0^t \frac{\partial\omega(r,s)}{\partial s} exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] ds;\ 0<\alpha<1, \tag{10}$$

satisfies the following useful properties:

$$\lim_{\alpha\to 1}[^{CF}D_t^{\alpha}\omega(r,t)] = \frac{\partial\omega(r,t)}{\partial t},\ L\left\{^{CF}D_t^{\alpha}\omega(r,t)\right\} = \frac{q\overline{\omega}(r,q)-\omega(r,0)}{(1-\alpha)q+\alpha}, \tag{11}$$

where $\overline{\omega}(r,q) = L[\omega(r,t)]$ is the Laplace transform of $\omega(r,t)$ and q is the transform parameter.

3 Solution of the Problem

In order to solve the fractional order partial differential equation (9) with the conditions (3) and $(7)_2$, we shall use the Laplace and Hankel transforms. Consequently, applying the Laplace transform to Eq. (9) and using the property $(11)_2$, we find that

$$\frac{\beta q}{q+\alpha\beta}\overline{\omega}(r,q) - \frac{\beta}{q+\alpha\beta}\frac{1}{r} = \nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\overline{\omega}(r,q);\ r>0, \tag{12}$$

where $\beta = 1/(1-\alpha)$. Now, multiplying Eq. (12) by $rJ_1(\rho r)$, integrating with respect to r from 0 to infinity and bearing in mind the usual hypotheses:

$$\lim_{r\to 0} r\overline{\omega}(r,q) = 0,\ \lim_{r\to\infty} r\overline{\omega}(r,q) < \infty;\ r\frac{\partial\overline{\omega}(r,q)}{\partial r}|_{r=0} < \infty,\ r\frac{\partial\overline{\omega}(r,q)}{\partial r}|_{r\to\infty} < \infty, \tag{13}$$

and the relations (A1)-(A3) from Appendix, it results that

$$\frac{\beta q}{q+\alpha\beta}\overline{\omega}_H(\rho,q) - \frac{\beta}{q+\alpha\beta}\frac{1}{\rho} = -\nu\rho^2\overline{\omega}_H(\rho,q), \tag{14}$$

where the Hankel transform $\overline{\omega}_H(\rho,q)$ of $\overline{\omega}(r,q)$ is defined by Eq. $(A3)_2$.

The solution of Eq. (14) can be written in the form

$$\overline{\omega}_H(\rho,q) = \frac{\beta}{\rho(\beta+\nu\rho^2)}.\frac{1}{q+\frac{\alpha\beta\nu\rho^2}{\beta+\nu\rho^2}} \tag{15}$$

and its inverse Laplace transform is

$$\omega_H(\rho,t) = \frac{\beta}{\rho(\beta+\nu\rho^2)} exp\left(-\frac{\alpha\beta\nu\rho^2}{\beta+\nu\rho^2}t\right). \tag{16}$$

Applying the inverse Hankel transform to this last equality, we find the dimensionless velocity field under the form

$$\omega(r,t) = \frac{\beta}{\nu}\int_0^\infty \frac{J_1(\rho r)}{\rho^2+\frac{\beta}{\nu}} exp\left(-\frac{\alpha\beta\rho^2}{\rho^2+\frac{\beta}{\nu}}t\right) d\rho. \tag{17}$$

The value of the non-dimensional circulation $\Gamma(r,t)$ on a circle of radius r, as it results from Eqs. $(8)_1$ and (17), is

$$\Gamma(r,t) = \frac{\beta r}{\nu}\int_0^\infty \frac{J_1(\rho r)}{\rho^2+\frac{\beta}{\nu}} exp\left(-\frac{\alpha\beta\rho^2}{\rho^2+\frac{\beta}{\nu}}t\right) d\rho. \tag{18}$$

Introducing Eq. (17) in $(8)_2$ we find the corresponding non-trivial shear stress $\tau(r,t)$ under the form

$$\tau(r,t) = -\frac{1}{2\pi}\frac{\beta}{\nu}\int_0^\infty \frac{\rho J_2(\rho r)}{\rho^2+\frac{\beta}{\nu}} exp\left(-\frac{\alpha\beta\rho^2}{\rho^2+\frac{\beta}{\nu}}t\right) d\rho. \tag{19}$$

Finally, it is worth pointing out the fact that making the fractional parameter $\alpha \to 1$ into Eqs. (17) and (18) and using Eq. (A4) from Appendix, we find the simple expressions

$$\omega(r,t) = \frac{1}{r}\left[1 - exp\left(-\frac{r^2}{4\nu t}\right)\right],\ \Gamma(r,t) = 1 - exp\left(-\frac{r^2}{4\nu t}\right), \tag{20}$$

which are just the dimensionless forms of the classical solutions (1) and (4).

Moreover, taking the limit of Eq. (19) when $\alpha \to 1$, we find that

$$\tau(r,t) = -\frac{1}{2\pi}\int_0^\infty \rho J_2(\rho r) e^{-\nu\rho^2 t} d\rho. \tag{21}$$

Now, based on the equality (A5) from appendix [13, Eq. 6.631], we can write the shear stress in the simpler form

$$\tau(r,t) = -\frac{1}{2\pi}\left(\frac{r}{4\nu t}\right)^2 F\left(2,3;-\frac{r^2}{4\nu t}\right), \tag{22}$$

where the degenerate confluent hypergeometric function $F(a,b;z)$ is defined by

$$F(a,b;z) = 1 + \frac{a}{b.1!}z + \frac{a(a+1)}{b(b+1)2!}z^2 + \frac{a(a+1)(a+2)}{b(b+1)(b+2)3!}z^3 + ... \tag{23}$$

Introducing Eq. (A6) in (22), the shear stress $\tau(r,t)$ takes the simple form

$$\tau(r,t) = \frac{1}{\pi r^2}\left\{\left(1+\frac{r^2}{4\nu t}\right) exp\left(-\frac{r^2}{4\nu t}\right) - 1\right\}, \tag{24}$$

which is the dimensionless form of Eq. (4.5) from [5].

Unfortunately, making $t = 0$ in Eq. (17) and using Eqs. (A3) and (A7) from Appendix, we find that

$$\omega(r,0) = \frac{1}{r} - \sqrt{\frac{\beta}{\nu}}K_1\left(r\sqrt{\frac{\beta}{\nu}}\right); \; r > 0, \tag{25}$$

where $K_1(\cdot)$ is the modified Bessel function of second kind and order one. Consequently, the initial condition $(7)_2$ is not satisfied and our solution seems to be wrong. In order to remove this mistrust we shall follow another way to show that the equality (25) is correctly determined.

For this, let us write Eq. (12) in an equivalent form

$$r^2\overline{\omega}'' + r\overline{\omega}' - (1 + a(q)r^2)\overline{\omega} = b(q)r, \tag{26}$$

where $a(q) = \frac{\beta}{\nu}\frac{q}{q+\alpha\beta}$, $b(q) = -\frac{\beta}{\nu}\frac{r}{q+\alpha\beta}$ and the notation " $'$ " is used for the partial derivative of $\overline{\omega}(r,t)$ with respect to r. A particular solution of nonhomogenous Bessel equation (26) is $\frac{1}{rq}$ and its general solution has the form

$$\overline{\omega}(r,q) = \frac{1}{rq} + C_1 I_1(r\sqrt{a(q)}) + C_2 K_1(r\sqrt{a(q)}), \tag{27}$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind and order one. According to the condition $(3)_1$, the constant C_1 has to be zero.

Bearing in mind the equality $(13)_1$ and the fact that

$$K_n(z) \approx \frac{(n-1)!}{2}\left(\frac{z}{2}\right)^{-n} \quad \text{for } z \leq n \; and \; n > 0 \tag{28}$$

it results that $C_2 = -\frac{\sqrt{a(q)}}{q}$ and

$$\overline{\omega}(r,q) = \frac{1}{rq} - \frac{\sqrt{a(q)}}{q}K_1\left(r\sqrt{a(q)}\right); \; r > o \tag{29}$$

or equivalently

$$q\overline{\omega}(r,q) = \frac{1}{r} - \sqrt{a(q)}K_1\left(r\sqrt{a(q)}\right);\ r > 0. \tag{30}$$

Taking the limit of Eq. (30) when $q \to \infty$ and using the property (A8) from Appendix, we obtain for $\omega(r,0)$ the same expression as that from Eq. (25). Consequently, the result is correct and it is not singular in the literature. Puri [14] studied the first problem of Stokes for Rivlin-Ericksen fluids and found a solution which does not satisfy the initial condition. Later, Bandelli et al. [3] and Bandelli and Rajagopal [15] showed that the Laplace transform method does not work for two different problems of second grade fluids because the obtained solutions do not satisfy the initial conditions. This is due to an incompatibility between the prescribed data.
A similar problem appears here because $\omega(r,0)$ tends to infinity for $r \to 0$. However, using the approximative evaluation (28) and the fact that

$$K_n(z) \approx \frac{\pi}{\sqrt{2\pi z}}e^{-z} \quad \text{for } z \gg n \quad \text{and } n > 0, \tag{31}$$

we can determine the magnitude of the deviation, namely

$$\omega(r,0) = \frac{1}{r} - \sqrt{\frac{\beta}{\nu}}K_1\left(r\sqrt{\frac{\beta}{\nu}}\right) \approx \begin{cases} 0, & \text{if } r\sqrt{\frac{\beta}{\nu}} \ll 1 \\ \frac{1}{r} - \sqrt{\frac{\pi}{2r}}\left(\frac{\beta}{\nu}\right)^{\frac{1}{4}} e^{-r\sqrt{\frac{\beta}{\nu}}}, & \text{if } r\sqrt{\frac{\beta}{\nu}} \gg 1. \end{cases} \tag{32}$$

For large values of $r\sqrt{\frac{\beta}{\nu}}$, the deviation from the initial condition is negligible. It tends to zero for $r \to \infty$.

Of course, a new exact solution for our problem can be obtained applying the inverse Laplace transform to Eq. (30). However, the inversion procedure is not always undemanding and often requires care and ingenuity. In this case the Stehfest's algorithm for numerical inversion of Laplace transforms [16] can be successfully used to get a numerical solution.

4 Numerical Results and Conclusions

In this note the time fractional Oseen problem is analytically studied using Laplace and Hankel transforms. Exact expressions for the dimensionless velocity $\omega(r,t)$ as well as for the corresponding circulation $\Gamma(r,t)$ on a circle of radius r and the non-trivial shear stress $\tau(r,t)$ are established in integral form in terms of Bessel function $J_1(\cdot)$. As it was to be expected, these expressions tend to the corresponding non-dimensional forms of classical solutions (20) and the solution (4.5) from [5] if the fractional parameter $\alpha \to 1$. The velocity field, given by Eq. (17), remains finite for $t > 0$ and $r \geq 0$ but it does not satisfy the initial condition $(7)_2$. Consequently it does not represent a smooth solution (cf. [3]) although is infinitely derivable in both variables. However, this is not a surprise, another similar example of velocity discontinuity at time $t = 0$ arises in the problem of a block mass m subjected to a blow P [17].

In order to get some physical insight of present results (17) and (18), the variations of the two entities of physical interest $\omega(r,t)$ and $\Gamma(r,t)$ with respect to the spatial and temporal variables r or t are presented in Figs. 1 and 2, respectively 3 and 4 for different values of the fractional parameter α. In all cases, as expected, the diagrams corresponding to the fractional model tend to superpose over those of classical solutions when the fractional parameter α tends to one. Fluids velocity, as it results from Fig. 1, increases up to a maximum value and then smoothly decreases to the asymptotic value for large value of r. The circulation $\Gamma(r,t)$, which describes the diffusion of vorticity, smoothly increases from zero value in $r = 0$ up to the asymptotic value 1(one) for r greater than 4(four). Both entities have greater values for fractional fluids in comparison to ordinary fluids.
Time variations of $\omega(r,t)$ and $\Gamma(r,t)$ are depicted in Figs. 3 and 4 for different values of r and the fractional parameter α. As form, the diagrams of the two entities are almost the same. For each r and α they smoothly decreases from a maximum at $t = 0$ to the zero value for increasing values of t. For small values of r (smaller than four), there exist a critical value of t up to which their magnitudes are higher for ordinary fluids. An opposite trend appears later and the diagrams corresponding to fractional fluid tend to superpose over those of ordinary fluid. The vortex, as expected, decreases in time and space. More exactly, it deaden in time and space.

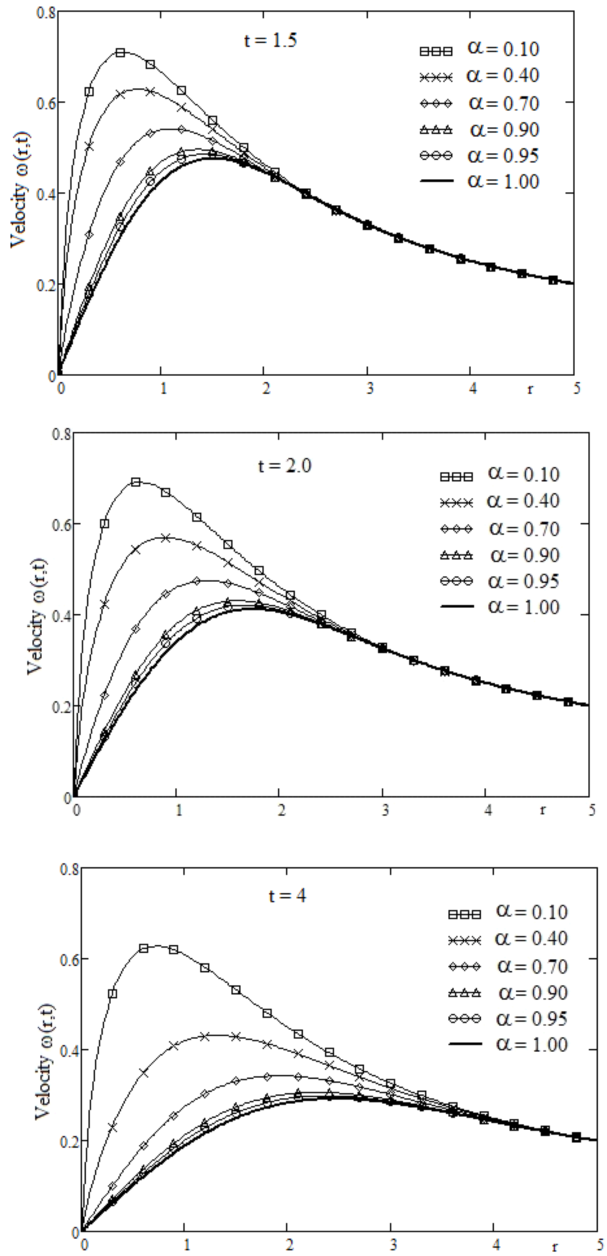

Figure 1: Profiles of the velocity $\omega(r,t)$ versus r, for $\nu = 0.3$ and different values of fractional parameter α and time t.

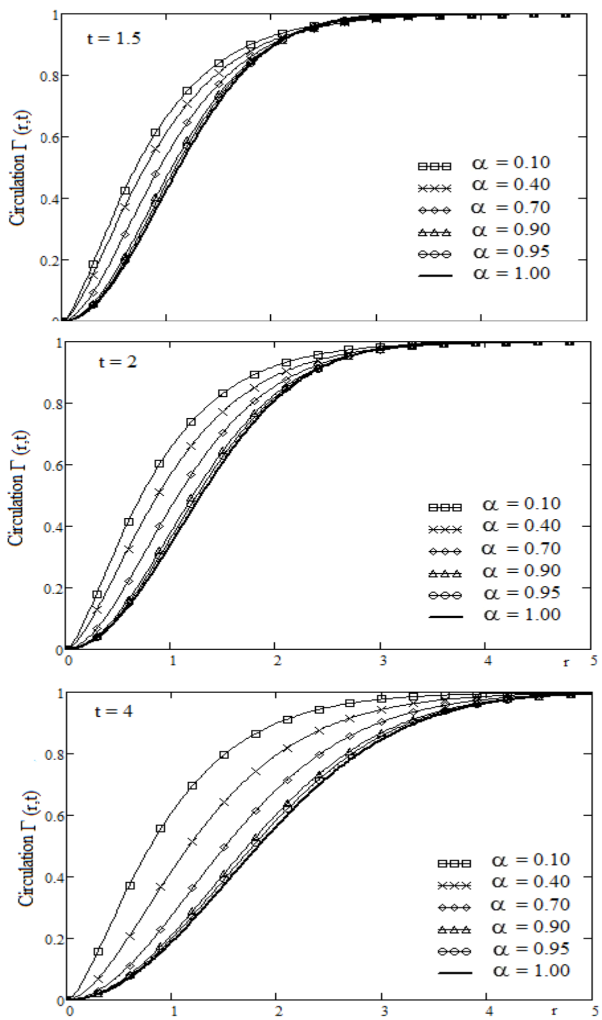

Figure 2: Profiles of the circulation $\Gamma(r,t)$ versus r, for $\nu = 0.3$ and different values of fractional parameter α and time t.

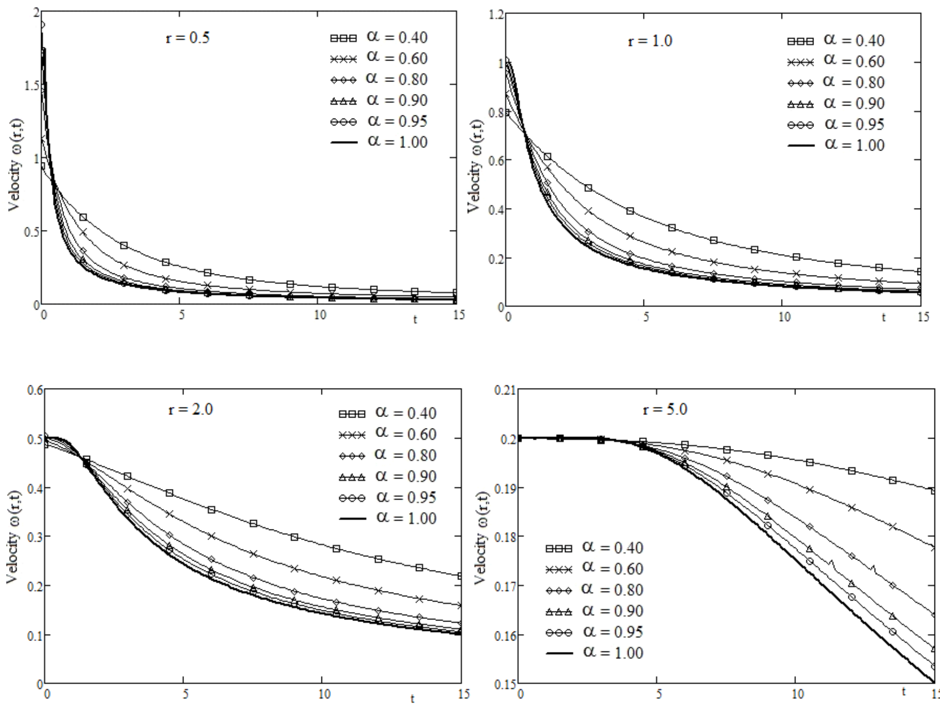

Figure 3: Profiles of the velocity $\omega(r,t)$ versus t, for $\nu = 0.3$ and different values of fractional parameter α and the spatial coordinate r.

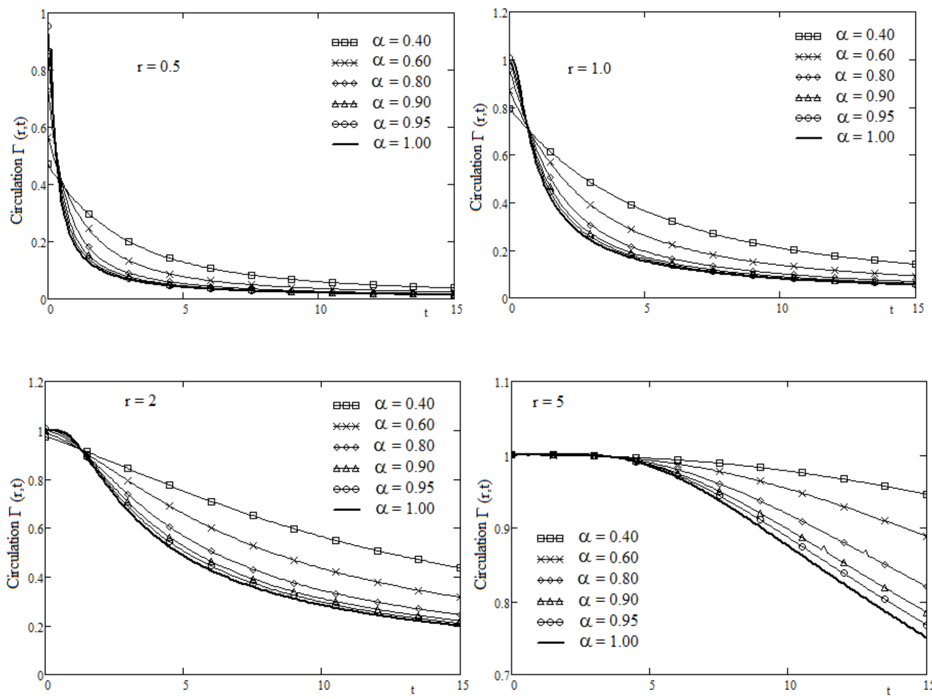

Figure 4: Profiles of the circulation $\Gamma(r,t)$ versus t, for $\nu = 0.3$ and different values of fractional parameter α and the spatial coordinate r.

Appendix

$$\int_0^\infty rJ_1(\rho r)(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2})\omega(r,t)dr=-\rho^2\omega_H(\rho,t). \qquad (A1)$$

$$J_\alpha(z)\approx\sqrt{\frac{2}{\pi z}}\cos(z-\frac{\alpha\pi}{2}-\frac{\pi}{4}) \quad \text{for large value of } z. \qquad (A2)$$

$$\int_0^\infty J_1(\rho r)dr=\frac{1}{\rho};\ \overline{\omega}_H(\rho,q)=\int_0^\infty r\overline{\omega}(r,q)J_1(\rho r)dr. \qquad (A3)$$

$$\int_0^\infty J_1(ax)e^{-b^2x^2}dx=\frac{1}{a}[1-exp(-\frac{a^2}{4b^2})];a\neq 0. \qquad (A4)$$

$$\int_0^\infty x^\mu J_\nu(ax)e^{-bx^2}dx=\frac{a^\nu\Gamma(\frac{\nu+\mu+1}{2})}{2^{\nu+1}b^{\frac{\nu+\mu+1}{2}}\Gamma(\nu+1)}F\left(\frac{\nu+\mu+1}{2},\,\nu+1;\,-\frac{a^2}{4b}\right). \qquad \text{(A5)}$$

$$x^2F(2,3;-x)=2[1-(1+x)e^{-x}]. \qquad (A6)$$

$$\int_0^\infty \frac{\rho^{\nu+1}J_\nu(\rho r)}{(\rho^2+a^2)^{\mu+1}}d\rho=\frac{a^{\nu-\mu}r^\mu}{2^\mu\Gamma(\mu+1)}K_{\nu-\mu}(ar). \qquad (A7)$$

$$\omega(r,0)=\lim_{q\to\infty}q\overline{\omega}(r,q). \qquad (A8)$$

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