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EXPONENTIAL STABILITY IN MEAN SQUARE OF A LARGE CLASS OF SINGULARLY PERTURBED STOCHASTIC LINEAR DIFFERENTIAL EQUATIONS *

Vasile Drăgan† Ioan-Lucian Popa‡ Hiroaki Mukaidani§
Toader Morozan¶

Dedicated to Professor Mihail Megan
on the occasion of his 70th anniversary

Abstract

A stability problem for a class of large-scale singularly perturbed stochastic systems (SPSSs) with state-multiplicative white noise and Markovian jumping parameters is considered. Based on the linear evolution operator, an exponential stability in mean square is investigated.

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†`Vasile.Dragan@imar.ro` Institute of Mathematics "Simion Stoilow" of the Romanian Academy P.O.Box 1-764, RO-014700, Bucharest, and the Academy of the Romanian Scientists, Romania;

‡`lucian.popa@uab.ro` Department of Exact Sciences and Engineering, "1 Decembrie 1918" University of Alba Iulia, 510009-Alba Iulia, Romania; This work was supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS/CCCDI - UEFISCDI, project number PN-III-P2-2.1-PED-2016-1835, within PNCDI III.

§`mukaida@hiroshima-u.ac.jp` Graduate School of Engineering, Hiroshima University, Higashi-Hiroshima, Japan

¶`Toader.Morozan@imar.ro` Institute of Mathematics "Simion Stoilow" of the Romanian Academy P.O.Box 1-764, RO-014700, Bucharest, Romania;

After introducing some preliminary results, it is shown that there exist small perturbation parameters that cause the original stochastic systems to be mean square stable by utilizing the stochastic Lyapunov differential equation. Moreover, the decay rate of the original SPSSs via the boundedness of the solution under the expectation operator are established explicitly.

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1 Introduction

The systems of differential equations with singular perturbations were intensively investigated in the last sixty years starting with the pioneered work of Tichonov [20]. We recall that a singularly perturbed system of differential equations contains small parameters as coefficients of the derivatives of some unknown functions of the system. Usually such small parameters are neglected, thus we may associated two subsystems of lower dimensions which are independent of the small parameters, namely the boundary layer subsystem (fast subsystem) and the reduced subsystem (slow subsystem), see e.g. [22] for the deterministic framework and [10, 21] for the stochastic case. Over the past decade, stabilization problems for singularly perturbed stochastic systems (SPSSs) have been deeply investigated. The sufficient conditions of stability based on a combination of the idea of the exponential stability of singularly perturbed stochastic systems have been established [18]. An application to the analysis of singularly perturbed Markov systems represented by random evolutions has been considered [16]. The problem of exponential stability of singularly perturbed systems with parametric white noise excitations has been studied in [12, 5] for the linear case and [18] for the nonlinear case. In general, it is important to evaluate the decay rate for the SPSS besides the stability. It is known that in the deterministic framework, there exist the reliable contributions in the analysis of the gap between the decay rate of the fast component and the decay rate of the slow component of the solutions of the system of exponentially stable singularly perturbed systems of differential equation. In contrast, such features of SPSSs with Markovian jumping parameters has not been fully developed.

In recent years, a stabilization problem for a class of singularly perturbed linear stochastic systems with state multiplicative white noise and Markovian jumping parameters was investigated [9]. It is worth pointing out that the Lyapunov type operator is a powerful tool to establish the sufficient condition that attains an exponential stability in mean square thus avoiding the difficulty of using the two-time-scale decomposition approach. However, time varying case is not discussed. Taking into consideration the fact that the stability analysis in SPSSs with Markovian jumping parameters has become a priority research topic, the investigation of the expectation behavior of their dynamics, when the perturbations parameters tend to zero, is extremely attractive.

In this paper, a stability problem for a class of singularly perturbed linear stochastic systems with state multiplicative white noise and Markovian jumping parameters is investigated. It should be noted that a set of sufficient condition for the exponential stability in the mean square sense of the stochastic differential equations with small perturbation parameters has not been derived exactly in the present time. Namely, there are few results for SPSSs with Markovian jumping parameters except for [9]. Therefore, this is one of the vital reasons that motivates us to investigate our current study.

This paper might be viewed as a time varying case of [9]. Because of the existence of small perturbation and Markovian jumping parameters, a linear evolution operator is used instead of the two-time scale decomposition technique. In the present work, it is shown that the same type of behaviour can be recovered in the stochastic case of the systems of singularly perturbed stochastic differential equations. Furthermore, the decay rate of SPSSs are established explicitly under some assumptions for the small perturbation parameters.

Notations: The notations used in this work are in general the standard ones. Here we recall only: $\mathbb{C}_\lambda$ the half plane of the form $\{z \in \mathbb{C} | \mathrm{Re}\ z < -\lambda\}$ and $\mathbb{C}_- = \{z \in \mathbb{C} | \mathrm{Re}\ z < 0\}$.

2 The problem

We consider the system of stochastic linear differential equations:

$$\begin{aligned} dx_1(t) &= [A_{11}(t,\eta_t)x_1(t) + A_{12}(t,\eta_t)x_2(t)]dt \\ &\quad + \sum_{k=1}^{r}[A_{k,11}(t,\eta_t)x_1(t) + A_{k,12}(t,\eta_t)x_2(t)]dw_k(t) \end{aligned} \tag{1a}$$

$$\begin{aligned} \varepsilon dx_2(t) &= [A_{21}(t,\eta_t)x_1(t) + A_{22}(t,\eta_t)x_2(t)]dt \\ &\quad + \mu\sum_{k=1}^{r}[A_{k,21}(t,\eta_t)x_1(t) + A_{k,22}(t,\eta_t)x_2(t)]dw_k(t) \end{aligned} \tag{1b}$$

where $t \in \mathbb{R}_+ = [0, \infty)$, $x_j(t) \in \mathbb{R}^{n_j}$, $j = 1, 2$, are the state variables (unknown functions) and $\varepsilon > 0$, $\mu > 0$ are small parameters often unknown.

In (1) $\{w(t)\}_{t\geq 0}$ $\left(w(t) = (w_1(t) \dots w_r(t))^T\right)$ is an $r-$dimensional standard Wiener process defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and $\{\eta_t\}_{t\geq 0}$ is a right continuous Markov process defined on the same probability space taking values in the finite set $\mathfrak{N} = \{1, 2, \dots, N\}$ and has the transition semigroup $P(t) = e^{Qt}$, $t \geq 0$, with $Q = (q_{ij}) \in \mathbb{R}^{N\times N}$. The elements q_{ij} of the generator matrix Q satisfy

$$q_{il} \geq 0 \ \text{ if } \ i \neq l, \ \ \sum_{j=1}^{N} q_{ij} = 0, \ \text{ for all } \ i, l \in \mathfrak{N}. \tag{2}$$

Throughout the paper we assume that $\{w(t)\}_{t\geq 0}$, $\{\eta_t\}_{t\geq 0}$ are independent stochastic processes and $\pi_0(i) \stackrel{\Delta}{=} \mathcal{P}(\eta_0 = i) > 0$, for all $i \in \mathfrak{N}$.
$\pi_0 = (\pi_0(1) \dots \pi_o(N))$ is the initial probability distribution of the Markov process. We shall write $A_{jl}(t, i)$ and $A_{k,jl}(t, i)$ anytime $\eta_t = i \in \mathfrak{N}$.

Regarding the coefficients of the system (1) we make the assumption:

H1) $t \longrightarrow A_{jl}(t, i) : \mathbb{R}_+ \to \mathbb{R}^{n_j \times n_l}$, $t \longrightarrow A_{k,jl}(t, i) : \mathbb{R}_+ \to \mathbb{R}^{n_j \times n_l}$ are bounded matrix valued functions that are globally Lipschitz continuous.

We set

$$A(t, i, \varepsilon) = \begin{bmatrix} A_{11}(t, i) & A_{12}(t, i) \\ \frac{1}{\varepsilon} A_{21}(t, i) & \frac{1}{\varepsilon} A_{22}(t, i) \end{bmatrix}$$
$$A_k(t, i, \varepsilon, \mu) = \begin{bmatrix} A_{k,11}(t, i) & A_{k,12}(t, i) \\ \frac{\mu}{\varepsilon} A_{k,21}(t, i) & \frac{\mu}{\varepsilon} A_{k,22}(t, i) \end{bmatrix}. \tag{3}$$

With these notations the system (1) can be written in a compact form, as:

$$dx(t) = A(t, \eta_t, \varepsilon)x(t)dt + \sum_{k=1}^{r} A_k(t, \eta_t, \varepsilon, \mu)x(t)dw_k(t) \tag{4}$$

where $x(t) = \left(x_1^T(t) \ \ x_2^T(t)\right)^T$. Since for each fixed $\varepsilon > 0$, $\mu > 0$, the system (4) is a stochastic differential equation (SDE) of type (1.22) from [7], then we may deduce from the developments from Section 1.12 from the aforementioned reference, that for each $t_0 \in \mathbb{R}_+$ and $x_0 = \left(x_{10}^T \ \ x_{20}^T\right)^T \in \mathbb{R}^{n_1+n_2}$, the system (1) has a unique solution

$$x(t; t_0, x_0, \varepsilon, \mu) = \left(x_1^T(t; t_0, x_0, \varepsilon, \mu) \ \ x_2^T(t; t_0, x_0, \varepsilon, \mu)\right)^T$$

starting from x_0 at the initial time t_0. According to the terminology used in the case of the deterministic singularly perturbed systems of differential equations $x_1(t; t_0, x_0, \varepsilon, \mu)$ will be named **slow component**, while $x_2(t; t_0, x_0, \varepsilon, \mu)$ will be called **fast component** of the solution $x(t; t_0, x_0, \varepsilon, \mu)$.

In this work we deal with the problem of exponential stability of the zero solution of a system of SDEs of type (1).

Definition 1. *We say that the zero solution of SDE* (1) *or, equivalently the SDE* (1) *is exponentially stable in mean square* (*ESMS*)*, if its solutions* $\left(x_1^T(t; t_0, x_0, \varepsilon, \mu) \;\; x_2^T(t; t_0, x_0, \varepsilon, \mu)\right)^T$ *satisfy*

$$\mathbb{E}[|x_1(t; t_0, x_0, \varepsilon, \mu)|^2 + |x_2(t; t_0, x_0, \varepsilon, \mu)|^2 \; |\eta_{t_0} = i] \leq \beta e^{-\alpha(t-t_0)}|x_0|^2 \quad (5)$$

for all $t \geq t_0 \geq 0$, $x_0 \in \mathbb{R}^{n_1+n_2}$, $i \in \mathfrak{N}$, *and arbitrary initial probability distribution* π_0 *of the Markov process, where* $\alpha > 0$, $\beta \geq 1$ *are constants not depending upon* t, t_0, x_0, π_0.

In this work $\mathbb{E}[\cdot|\eta_{t_0} = i]$ stands for the conditional expectation with respect to the event $\{\eta_{t_0} = i\}$.

Our goal is to provide a set of sufficient conditions not depending upon the small parameters $\varepsilon > 0$, $\mu > 0$ (often unknown) that guarantee the exponential stability in the mean square sense of the SDEs of type (1) for sufficient small values of the two parameters ε, and μ.

Since the coefficients of SDE (1) depend upon ε and μ it is expected that the decay rate α from (5) to be dependent upon these parameters. It is known that in the deterministic framework (see for example [14], [3], [4], [15]) there exists a deviation of order ε^{-1} between the decay rate of the fast component and the decay rate of the slow component of the solutions of the system of exponentially stable singularly perturbed differential equations.

In the present work, we shall show that the same type of behaviour can be recovered in the stochastic case of the systems of singularly perturbed SDEs of type (1). We shall see that in this case beside the parameters ε, μ an important role is played by the new parameter $\nu = \mu^2/\varepsilon$. In the special case $\mathfrak{N} = \{1\}$ (no Markovian switching) the result proved here recovers a part from the result of [5]. The results derived in this work extend to the time varying case the results from [9].

3 Some auxiliary results

In this section we shall include several results from the deterministic framework which will be used in the derivation of the main results of the present work.

Lemma 1. *Let $(\mathcal{X}, \|\cdot\|)$ be a finite dimensional Banach space and $\mathcal{B}(\mathcal{X})$ be the space of the linear operators defined on $\mathcal{X}$. Let $\mathcal{L} : \mathbb{R}_+ \longrightarrow \mathcal{B}(\mathcal{X})$ be a operator valued function with the properties:*

(a) *$t \longrightarrow \mathcal{L}(t)$ is a bounded and globally Lipschitz continuous function;*

(b) *for each $t \in \mathbb{R}_+$ the eigenvalues of the operator $\mathcal{L}(t)$ are located in the half plane $\mathbb{C}_\alpha$ where $\alpha > 0$ does not depend upon t.*

Let $\mathbf{T}(t, t_0, \varepsilon)$ be the linear evolution operator on $\mathcal{X}$ defined by the linear differential equation $\varepsilon\dot{\mathbf{x}}(t) = \mathcal{L}(t)\mathbf{x}(t)$. Under these conditions for each $\tilde{\alpha} \in (0, \alpha)$ there exists $\varepsilon_0 = \varepsilon_0(\tilde{\alpha}, \|\mathcal{L}\|_\infty) > 0$ such that

$$\|\mathbf{T}(t, t_0, \varepsilon)\| \leq \tilde{\beta} e^{\frac{-\tilde{\alpha}(t-t_0)}{\varepsilon}}, \ \ \forall\, t \geq t_0 \geq 0, \varepsilon \in (0, \tilde{\varepsilon}],$$

where $\tilde{\beta} \geq 1$ is a constant depending upon $\tilde{\alpha}$, $\|\mathcal{L}\|_\infty$, as well as the Lipschitz constant of $\mathcal{L}(\cdot)$.

The proof may be done following standard arguments in singular perturbation theory. For details, we refer to [3] and [11].

Now, we consider $(\mathcal{X}_k, \|\cdot\|_k)$, $k = 1, 2$ two finite dimensional Banach spaces. On $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ we consider the system of linear differential equations:

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= M_{11}(t, \varepsilon, \delta)\mathbf{x}_1(t) + M_{12}(t, \varepsilon, \delta)\mathbf{x}_2(t) \\ \varepsilon\dot{\mathbf{x}}_2(t) &= M_{21}(t, \varepsilon, \delta)\mathbf{x}_1(t) + M_{22}(t, \varepsilon, \delta)\mathbf{x}_2(t) \end{aligned} \tag{6}$$

where $(t, \varepsilon, \delta) \longrightarrow M_{jk}(t, \varepsilon, \delta) : \mathbb{R}_+ \times [0, 1) \times \mathbb{B}_\rho(\delta_0) \longrightarrow \mathcal{B}(\mathcal{X}_k, \mathcal{X}_j)$ are operator valued functions $j, k = 1, 2$, $\mathbb{B}_\rho(\delta_0) \subset \mathbb{R}^d$ is the ball of radius $\rho > 0$ centered in $\delta_0 \in \mathbb{R}^d$.

Proposition 1. *Assume:*

(a) *the operator valued functions $M_{jk}(\cdot, \cdot, \cdot)$ are bounded on their domain of definition and have the additional properties*

$$\|M_{jk}(t_1, \varepsilon, \delta) - M_{jk}(t_2, \varepsilon, \delta)\| \leq \gamma |t_1 - t_2|, \tag{7}$$

for all $t_1, t_2 \in \mathbb{R}_+$, $(\varepsilon, \delta) \in [0, 1) \times \mathbb{B}_\rho(\delta_0)$, $\gamma > 0$ being a constant,

$$\|M_{jk}(t, \varepsilon, \delta) - M_{jk}(t, 0, \delta_0)\| \leq \hat{\gamma}(\varepsilon + |\delta - \delta_0|) \tag{8}$$

for all $(t, \varepsilon, \delta) \in \mathbb{R}_+ \times [0, 1) \times \mathbb{B}_\rho(\delta_0)$, $\hat{\gamma} > 0$ being a constant and $|\cdot|$ is the Euclidian norm on $\mathbb{R}^d$, while $\|\cdot\|$ from (7) and (8) are the operator norms induced by $\|\cdot\|_1$ and $\|\cdot\|_2$;

(b) *for each* $t \in \mathbb{R}_+$, *the eigenvalues of the linear operator* $M_{22}(t,0,\delta_0)$ *are located in the half plane* $\mathbb{C}_{2\alpha_f}$ *where* $\alpha_f > 0$ *does not depend upon* t;

(c) *the linear evolution operator* $\mathbf{T}_s(t,t_0)$ *defined by the linear differential equation on* $\mathcal{X}_1$:

$$\dot{\mathbf{x}}_1(t) = M_s(t)\mathbf{x}_1(t)$$

satisfies

$$\|\mathbf{T}_s(t,t_0)\| \le \beta_s e^{-2\alpha_s(t-t_0)} \tag{9}$$

for all $t \ge t_0 \ge 0$, $\beta_s \ge 1$, $\alpha_s > 0$ *do not depend by* t *and* t_0 *and*

$$M_s(t) = M_{11}(t,0,\delta_0) - M_{12}(t,0,\delta_0)M_{22}^{-1}(t,0,\delta_0)M_{21}(t,0,\delta_0).$$

Under these conditions there exist $\varepsilon^* \in (0,1]$, $\rho^* \in (0,\rho]$ *such that the system* (6) *is exponentially stable for any* $(\varepsilon,\delta) \in (0,\varepsilon^*] \times \mathbb{B}_{\rho^*}(\delta_0)$. *Moreover, the solutions of system* (6) *have the upper bounds of the form*

$$\|\mathbf{x}_1(t,\varepsilon,\delta)\|_1 \le \beta_1 e^{-\alpha_s(t-t_0)}\left(\|\mathbf{x}_1(t_0,\varepsilon,\delta)\|_1 + \varepsilon\|\mathbf{x}_2(t_0,\varepsilon,\delta)\|_2\right) \tag{10a}$$

$$\begin{aligned}\|\mathbf{x}_2(t,\varepsilon,\delta)\|_2 \le \beta_2 \Big[& e^{-\frac{\alpha_f(t-t_0)}{\varepsilon}}\|\mathbf{x}_2(t_0,\varepsilon,\delta)\|_2 + e^{-\alpha_s(t-t_0)}(\|\mathbf{x}_1(t_0,\varepsilon,\delta)\|_1 \\ & + \varepsilon\|\mathbf{x}_2(t_0,\varepsilon,\delta)\|_2)\Big]\end{aligned} \tag{10b}$$

for all $t \ge t_0 \ge 0$, $\beta_k \ge 1$, $k = 1,2$, *being constants independent of* $t, t_0, \varepsilon, \delta$.

Proof. Let $\mathbf{T}(t,t_0;\varepsilon,\delta)$ and $\mathbb{T}(t,t_0;\varepsilon)$ be the linear evolution operators on $\mathcal{X}_2$ defined by the linear differential equations

$$\varepsilon\dot{\mathbf{x}}_2(t) = M_{22}(t,\varepsilon,\delta)\mathbf{x}_2(t)$$

and

$$\varepsilon\dot{\mathbf{x}}_2(t) = M_{22}(t,0,\delta_0)\mathbf{x}_2(t),$$

respectively. Applying Lemma 1 for $\tilde{\alpha} = \frac{3}{2}\alpha_f$ we deduce that there exists $\varepsilon_1 > 0$ such that

$$\|\mathbb{T}(t,t_0;\varepsilon)\| \le \xi_1 e^{-\frac{3\alpha_f(t-t_0)}{2\varepsilon}}$$

for all $t \ge t_0 \ge 0$, $0 < \varepsilon \le \varepsilon_1$. Further we write

$$\begin{aligned}\mathbf{T}(t,t_0;\varepsilon,\delta)\mathbf{x} &= \mathbb{T}(t,t_0;\varepsilon)\mathbf{x} \\ &+ \frac{1}{\varepsilon}\int_{t_0}^{t} \mathbb{T}(t,\tau;\varepsilon)\left(M_{22}(\tau,\varepsilon,\delta) - M_{22}(\tau,0,\delta_0)\right)\mathbf{T}(\tau,t_0;\varepsilon,\delta)\mathbf{x}d\tau\end{aligned} \tag{11}$$

Employing (8) and (11) we get

$$\|\mathbf{T}(t,t_0;\varepsilon,\delta)\mathbf{x}\|_2 \le \xi_1\|\mathbf{x}\|_2 e^{-\frac{3\alpha_f(t-t_0)}{2\varepsilon}} + \frac{1}{\varepsilon}\xi_1\hat{\gamma}\left(\varepsilon+|\delta-\delta_0|\right) \times \int_{t_0}^{t} e^{-\frac{3\alpha_f(t-\tau)}{2\varepsilon}}\|\mathbf{T}(\tau,t_0;\varepsilon,\delta)\mathbf{x}\|_2 d\tau.$$

Setting $\psi(t,\varepsilon,\delta) = \sup_{t_0\le\tau\le t}\|\mathbf{T}(\tau,t_0;\varepsilon,\delta)\mathbf{x}\|_2$ we obtain

$$\left[1-\gamma_1(\varepsilon+|\delta-\delta_0|)\right]\psi(t,\varepsilon,\delta) \le \xi_1\|\mathbf{x}\|_2 e^{-\frac{3\alpha_f(t-t_0)}{2\varepsilon}}$$

for all $t\ge t_0\ge 0$, $\mathbf{x}\in\mathcal{X}_2$ where $\gamma_1 = 2\hat{\gamma}\xi_1\frac{1}{3\alpha_f}$. Choose $c\in(0,1)$ fixed. If $\varepsilon+|\delta-\delta_0|\le\frac{1-c}{\xi_1}$ we may conclude that

$$\|\mathbf{T}(t,t_0;\varepsilon,\delta)\mathbf{x}\|_2 \le \xi_1\frac{1}{c}e^{-\frac{3\alpha_f(t-t_0)}{2\varepsilon}}\|\mathbf{x}\|_2 \tag{12}$$

for all $t\ge t_0\ge 0$, $\mathbf{x}\in\mathcal{X}_2$.

Further, the assumption (b) together with (8) allow us to deduce that there exist $\tilde{\varepsilon}_1>0$, $\tilde{\rho}_1>0$ such that $M_{22}(t,\varepsilon,\delta)$ is invertible and, additionally $(t,\varepsilon,\delta)\longrightarrow M_{22}^{-1}(t,\varepsilon,\delta)$ is bounded for $t\in\mathbb{R}_+$, $0\le\varepsilon\le\tilde{\varepsilon}_1$ and $|\delta-\delta_0|\le\tilde{\rho}_1$. We set

$$M_s(t,\varepsilon,\delta) \stackrel{\Delta}{=} M_{11}(t,\varepsilon,\delta) - M_{12}(t,\varepsilon,\delta)M_{22}^{-1}(t,\varepsilon,\delta)M_{21}(t,\varepsilon,\delta).$$

One shows that

$$\|M_s(t,\varepsilon,\delta)-M_s(t)\| \le \hat{\gamma}_1(\varepsilon+|\delta-\delta_0|)$$

for all $t\in\mathbb{R}_+$ and ε, δ such that $\varepsilon+|\delta-\delta_0|$ is sufficiently small.

Let $\mathbf{T}_s(t,t_0;\varepsilon,\delta)$ be the linear evolution operator defined on $\mathcal{X}_1$ by the linear differential equation

$$\dot{\mathbf{x}}_1(t) = M_s(t,\varepsilon,\delta)\mathbf{x}_1(t).$$

Employing (9) and Gronwall's Lemma one shows that

$$\|\mathbf{T}(t,t_0;\varepsilon,\delta)\| \le \hat{\beta}_s e^{-\frac{3\alpha_s(t-t_0)}{2}} \tag{13}$$

for all $t \geq t_0 \geq 0$, $\varepsilon > 0$, $\delta \in \mathbb{B}_\rho(\delta_0)$ are such that $\varepsilon + |\delta - \delta_0|$ is sufficiently small. The solution of the system (6) has the representation formulae:

$$\begin{aligned}\mathbf{x}_1(t,\varepsilon,\delta) &= \mathbf{T}_s(t,t_0;\varepsilon,\delta)\mathbf{x}_1(t_0,\varepsilon,\delta) + \int_{t_0}^{t} \mathbf{T}_s(t,\tau;\varepsilon,\delta)M_{12}(\tau,\varepsilon,\delta) \\ &\quad \times \left(\mathbf{x}_2(\tau,\varepsilon,\delta) + M_{22}^{-1}(\tau,\varepsilon,\delta)M_{21}(\tau,\varepsilon,\delta)\mathbf{x}_1(\tau,\varepsilon,\delta)\right) d\tau, \qquad (14a)\\ \mathbf{x}_2(t,\varepsilon,\delta) &= \mathbf{T}(t,t_0;\varepsilon,\delta)\mathbf{x}_2(t_0,\varepsilon,\delta) \\ &\quad + \frac{1}{\varepsilon}\int_{t_0}^{t} \mathbf{T}(t,\tau;\varepsilon,\delta)M_{21}(\tau,\varepsilon,\delta)\mathbf{x}_1(\tau,\varepsilon,\delta)d\tau. \qquad (14b)\end{aligned}$$

Substituting (14b) in (14a) and using standard techniques from singular perturbation theory, one obtains via (7), (8), (12) and (13) that the slow component $\mathbf{x}_1(t,\varepsilon,\delta)$ of the solution of the system (6) has an asymptotic behaviour of the form described by the first inequality in (10). Finally, from (13), (14b) together with the upper bound of $\|\mathbf{x}_1(\tau,\varepsilon,\delta)\|_1$ given by the first inequality in (10) we may obtain that the fast component $\mathbf{x}_2(t,\varepsilon,\delta)$ of the solution of (6) has the asymptotic behaviour of the form described by the second inequality from (10). Thus the proof is complete. □

4 The main result

Let $x(t) = (x_1^T(t,t_0,x_0;\varepsilon,\mu) \;\; x_2^T(t,t_0,x_0;\varepsilon,\mu))^T$ be the solution of the system (1) starting from $x_0 = (x_{10}^T \;\; x_{20}^T)^T \in \mathbb{R}^{n_1+n_2}$ at the initial time $t_0 \geq 0$. We set $X(t,i) = \mathbb{E}[x(t)x^T(t)\chi_{\{\eta_t=i\}}]$, $t \geq t_0$, $i \in \mathfrak{N}$, $\chi_{\{\eta_t=i\}}$ is the indicator function of the event $\{\eta_t = i\}$. We set $n = n_1 + n_2$ and $\mathcal{S}_n \subset \mathbb{R}^{n\times n}$ stands for the linear space of the symmetric matrices of size $n \times n$. We denote $\mathcal{S}_n^N = \mathcal{S}_n \times \mathcal{S}_n \times \ldots \times \mathcal{S}_n$. Applying Theorem 3.1.6 from [7] we obtain that $t \longrightarrow (X(t,1), X(t,2), \ldots, X(t,N))$ is the solution of the following problem with given initial values on the space $\mathcal{S}_n^N$:

$$\begin{aligned}\dot{X}(t,i) &= A(t,i,\varepsilon)X(t,i) + X(t,i)A^T(t,i,\varepsilon) \\ &\quad + \sum_{k=1}^{r} A_k(t,i,\varepsilon,\mu)X(t,i)A_k^T(t,i,\varepsilon,\mu) \\ &\quad + \sum_{j=1}^{N} q_{ji}X(t,j), \quad i \in \mathfrak{N} \qquad (15)\end{aligned}$$

$X(t_0, l) = x_0 x_0^T \pi_{t_0}(l)$, $l \in \mathfrak{N}$, where $\pi_{t_0} = (\pi_{t_0}(1), \dots, \pi_{t_0}(N))$ is the probability distribution of the random variable η_{t_0}, i.e. $\pi_{t_0}(l) = \mathcal{P}\{\eta_{t_0} = l\}$, $l \in \mathfrak{N}$. Let

$$\begin{bmatrix} X_{11}(t,i) & X_{12}(t,i) \\ X_{12}^T(t,i) & X_{22}(t,i) \end{bmatrix}$$

be the partition of $X(t,i)$ compatible with the partition of the coefficients of (15) given in (3). By direct calculation one obtains the following partition of (15):

$$\begin{aligned} \dot{X}_{11}(t,i) &= A_{11}(t,i)X_{11}(t,i) + A_{12}(t,i)X_{12}^T(t,i) + X_{11}(t,i)A_{11}^T(t,i) \\ &\quad + X_{12}(t,i)A_{12}^T(t,i) + \sum_{k=1}^{r}[A_{k,11}(t,i)X_{11}(t,i)A_{k,11}^T(t,i) \\ &\quad + A_{k,12}(t,i)X_{12}^T(t,i)A_{k,11}^T(t,i) + A_{k,11}(t,i)X_{12}(t,i)A_{k,12}^T(t,i) \\ &\quad + A_{k,12}(t,i)X_{22}(t,i)A_{k,12}^T(t,i)] + \sum_{j=1}^{N} q_{ji}X_{11}(t,j), \qquad (16a) \\ \varepsilon\dot{X}_{12}(t,i) &= \varepsilon A_{11}(t,i)X_{12}(t,i) + \varepsilon A_{12}(t,i)X_{22}(t,i) + X_{11}(t,i)A_{21}^T(t,i) \\ &\quad + X_{12}(t,i)A_{22}^T(t,i) + \mu\sum_{k=1}^{r} t[A_{k,11}(t,i)X_{11}(t,i)A_{k,21}^T(t,i) \\ &\quad + A_{k,12}(t,i)X_{12}^T(t,i)A_{k,21}^T(t,i) + A_{k,11}(t,i)X_{12}(t,i)A_{k,22}^T(t,i) \\ &\quad + A_{k,12}(t,i)X_{22}(t,i)A_{k,22}^T(t,i)] + \varepsilon\sum_{j=1}^{N} q_{ji}X_{12}(t,j), \qquad (16b) \\ \varepsilon\dot{X}_{22}(t,i) &= A_{21}(t,i)X_{12}(t,i) + A_{22}(t,i)X_{22}(t,i) + X_{12}^T(t,i)A_{21}^T(t,i) \\ &\quad + X_{22}(t,i)A_{22}^T(t,i) + \frac{\mu^2}{\varepsilon}\sum_{k=1}^{r}[A_{k,21}(t,i)X_{11}(t,i)A_{k,21}^T(t,i) \\ &\quad + A_{k,22}(t,i)X_{12}^T(t,i)A_{k,21}^T(t,i) + A_{k,21}(t,i)X_{12}(t,i)A_{k,22}^T(t,i) \\ &\quad + A_{k,22}(t,i)X_{22}(t,i)A_{k,22}^T(t,i)] + \varepsilon\sum_{j=1}^{N} q_{ji}X_{22}(t,j) \qquad (16c) \\ X_{jl}(t_0,i) &= x_{0j}x_{0l}^T\pi_{t_0}(i), \quad j,l = 1,2, \quad i \in \mathfrak{N}. \qquad (16d) \end{aligned}$$

Further on, we shall show that the system of linear differential equations (16) can be regarded as a system of singularly perturbed linear differential equations of the form (6). To this end, we take $\mathcal{X}_1 = \mathcal{S}_{n_1}^N$. The elements $\mathbf{X}_1$ are finite sequences of symmetric matrices of size $n_1 \times n_1$, that is, $\mathbf{X}_1 =$

$(X_1(1), \ldots, X_1(N))$. On the space $\mathcal{X}_1$ we introduce the norm $\|\cdot\|_1$ defined by

$$\|\mathbf{X}_1\|_1 = \max_{i \in \mathfrak{N}} |X_1(i)| \tag{17}$$

for all $\mathbf{X}_1 \in \mathcal{X}_1$, $|X_1(i)|$ being the spectral norm of the matrix $X_1(i)$.
Also, we set

$$\mathcal{X}_2 \triangleq \underbrace{\left(\mathbb{R}^{n_1 \times n_1} \times \mathcal{S}_{n_2}\right) \times \ldots \times \left(\mathbb{R}^{n_1 \times n_2} \times \mathcal{S}_{n_2}\right)}_{N \text{ times}}.$$

Its elements $\mathbf{X}_2$ are finite sequences of the form

$$\mathbf{X}_2 = \left((X_{12}(1), X_{22}(1)), \ldots, (X_{12}(N), X_{22}(N))\right).$$

On the space $\mathcal{X}_2$ we introduce the norm

$$\|\mathbf{X}_2\|_2 = \max_{i \in \mathfrak{N}} \left\{\max\left(|X_{12}(i)|, |X_{22}(i)|\right)\right\}. \tag{18}$$

One can see that $(\mathcal{X}_j, \|\cdot\|_j)$, $j = 1, 2$ are finite dimensional Banach spaces. The system (16) may be regarded as a system of singularly perturbed linear differential equations on the space $\mathcal{X}_1 \times \mathcal{X}_2$:

$$\dot{\mathbf{X}}_1(t) = \mathbb{M}_{11}(t, \varepsilon, \mu, \nu)[\mathbf{X}_1(t)] + \mathbb{M}_{12}(t, \varepsilon, \mu, \nu)[\mathbf{X}_2(t)] \tag{19a}$$

$$\varepsilon \dot{\mathbf{X}}_2(t) = \mathbb{M}_{21}(t, \varepsilon, \mu, \nu)[\mathbf{X}_1(t)] + \mathbb{M}_{22}(t, \varepsilon, \mu, \nu)[\mathbf{X}_2(t)] \tag{19b}$$

where $\mathbf{X}_l \longrightarrow \mathbb{M}_{jl}(t, \varepsilon, \mu, \nu)[\mathbf{X}_l] : \mathcal{X}_l \longrightarrow \mathcal{X}_j$ are defined as follows:

$$\begin{aligned} &\mathbb{M}_{11}(t, \varepsilon, \mu, \nu)[\mathbf{X}_1](i) \\ &\quad \triangleq A_{11}(t, i) X_{11}(i) + X_{11}(i) A_{11}^T(t, i) \\ &\qquad + \sum_{k=1}^{r} A_{k,11}(t, i) X_{11}(i) A_{k,11}^T(t, i) + \sum_{j=1}^{N} q_{ji} X_{11}(j) \end{aligned} \tag{20}$$

$i \in \mathfrak{N}$, for all $\mathbf{X}_1 = (X_{11}(1), \ldots, X_{11}(N)) \in \mathcal{X}_1$.

$$\begin{aligned} &\mathbb{M}_{12}(t, \varepsilon, \mu, \nu)[\mathbf{X}_2](i) \\ &\quad \triangleq A_{12}(t, i) X_{12}^T(i) + X_{12}(i) A_{12}^T(t, i) \\ &\qquad + \sum_{k=1}^{r} \left[A_{k,12}(t, i) X_{12}^T(i) A_{k,11}^T(t, i) + A_{k,11}(t, i) X_{12}(i) A_{k,12}^T(t, i)\right. \\ &\qquad \left. + A_{k,12}(t, i) X_{22}(i) A_{k,12}^T(t, i)\right] \end{aligned} \tag{21}$$

$i \in \mathfrak{N}$, for all $\mathbf{X}_2 = ((X_{12}(1), X_{22}(1)), \ldots, (X_{12}(N), X_{22}(N))) \in \mathcal{X}_2$.

Furthermore, we have

$$\mathbb{M}_{21}(t, \varepsilon, \mu, \nu)[\mathbf{X}_1](i) \stackrel{\Delta}{=} \left(\mathbb{M}_{21}^1(t, \varepsilon, \mu, \nu)[\mathbf{X}_1](i), \mathbb{M}_{21}^2(t, \varepsilon, \mu, \nu)[\mathbf{X}_1](i)\right), \quad (22)$$

where

$$\begin{aligned}\mathbb{M}_{21}^1(t, \varepsilon, \mu, \nu)[\mathbf{X}_1](i) &= X_{11}(i) A_{21}^T(t, i) \\ &+ \mu \sum_{k=1}^{r} A_{k,11}(t, i) X_{11}(i) A_{k,21}^T(t, i), \end{aligned} \quad (23a)$$

$$\mathbb{M}_{21}^2(t, \varepsilon, \mu, \nu)[\mathbf{X}_1](i) = \nu \sum_{k=1}^{r} A_{k,21}(t, i) X_{11}(i) A_{k,21}^T(t, i). \quad (23b)$$

On the other hand, we also have

$$\mathbb{M}_{22}(t, \varepsilon, \mu, \nu)[\mathbf{X}_2](i) \stackrel{\Delta}{=} \left(\mathbb{M}_{22}^1(t, \varepsilon, \mu, \nu)[\mathbf{X}_2](i), \mathbb{M}_{22}^2(t, \varepsilon, \mu, \nu)[\mathbf{X}_2](i)\right), \quad (24)$$

where

$$\begin{aligned}\mathbb{M}_{22}^1(t, \varepsilon, \mu, \nu)[\mathbf{X}_2](i) &= \varepsilon A_{11}(t, i) X_{12}(i) + \varepsilon A_{12}(t, i) X_{22}(i) + X_{12}(i) A_{22}^T(t, i) \\ &+ \mu \sum_{k=1}^{r} \left[A_{k,12}(t, i) X_{12}^T(i) A_{k,21}^T(t, i) + A_{k,11}(t, i) X_{12}(i) A_{k,22}^T(t, i) \right. \\ &\left. + A_{k,12}(t, i) X_{22}(i) A_{k,22}^T(t, i)\right] + \varepsilon \sum_{j=1}^{N} q_{ji} X_{12}(j), \end{aligned} \quad (25a)$$

$$\begin{aligned}\mathbb{M}_{22}^2(t, \varepsilon, \mu, \nu)[\mathbf{X}_2](i) &= A_{21}(t, i) X_{12}(i) + A_{22}(t, i) X_{22}(i) + X_{12}^T(i) A_{21}^T(t, i) \\ &+ X_{22}(i) A_{22}^T(t, i) + \nu \sum_{k=1}^{r} \left[A_{k,22}(t, i) X_{12}^T(t, i) A_{k,21}^T(t, i) \right. \\ &\left. + A_{k,21}(t, i) X_{12}(i) A_{k,22}^T(t, i) + A_{k,22}(t, i) X_{22}(i) A_{k,22}^T(t, i)\right] \\ &+ \varepsilon \sum_{j=1}^{N} q_{ji} X_{22}(j) \end{aligned} \quad (25b)$$

for all $\mathbf{X}_2 = ((X_{12}(1), X_{22}(1)), \ldots, (X_{12}(N), X_{22}(N))) \in \mathcal{X}_2$.

Comparing (19)-(24) with (16) we remark the natural occurrence of the quantity $\nu = \mu^2/\varepsilon$. This quantity must be interpreted as an additional parameter of the problem under investigation. We note that even if $\varepsilon \longrightarrow 0_+$, $\mu \longrightarrow 0_+$, the limit $\lim\limits_{(\varepsilon,\mu)\to(0_+,0_+)} \mu^2/\varepsilon$ does not exist. That is why, in the following we are making the assumption:

H2) The values of the small parameters $\varepsilon > 0$, $\mu > 0$ are such that the values of μ^2/ε tend to a nominal value ν_0.

Remark 1. *The assumption* **H2**) *is fulfilled if, for example,* $\mu = \psi(\varepsilon)$, *where* $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *is a function with the properties:*

(a) $\psi(\varepsilon) = 0$ *if and only if* $\varepsilon = 0$;

(b) $\lim\limits_{\varepsilon \to 0_+} \psi(\varepsilon) = 0$;

(c) $\lim\limits_{\varepsilon \to 0_+} \frac{\psi^2(\varepsilon)}{\varepsilon} = \nu_0$.

In many works was used $\psi(\varepsilon) = \varepsilon^\theta$ *with* $\theta \geq \frac{1}{2}$ (*see for example* [2], [5], [6], [8], [13], [17], [18] *and the references therein*).

In applications, the value of the nominal parameter ν_0 *is established together with the mathematical model of the phenomena under investigation.*

Let us remark that if we take $\delta = (\mu, \nu)$ and $\delta_0 = (0, \nu_0)$, the system (19) is of type (6). Hence, we may apply the result from Proposition 1 to obtain a set of sufficient conditions that guarantee the exponential stability of the system (19) for any $\varepsilon > 0$, $\mu > 0$ small enough.

Proposition 2. *Assume:*

(a) *the assumptions* **H1**) *and* **H2**) *are fulfilled;*

(b) *for any* $t \in \mathbb{R}_+$, *the eigenvalues of the linear operators* $\mathbb{M}_{22}(t, 0, 0, \nu_0) : \mathcal{X}_2 \to \mathcal{X}_2$ *are located in the half plane* $\mathbb{C}_{2\alpha_f}$ *for some* $\alpha_f > 0$ *not depending upon* t;

(c) *the linear evolution operator* $\mathbf{T}_s(t, t_0)$ *defined by the linear differential equation on* $\mathcal{X}_1$:
$$\dot{\mathbf{X}}_1(t) = \mathbb{M}_s(t)[\mathbf{X}_1(t)]$$
satisfies
$$\|\mathbf{T}_s(t, t_0)\| \leq \beta_s e^{-2\alpha_s(t-t_0)} \tag{26}$$
for all $t \geq t_0 \geq 0$, $\beta_s \geq 1$, $\alpha_s > 0$, *not depending upon* t, t_0, *and,* $\mathbb{M}_s(t) : \mathcal{X}_1 \to \mathcal{X}_1$ *being defined by*
$$\mathbb{M}_s(t) = \mathbb{M}_{11}(t, 0, 0, \nu_0) - \mathbb{M}_{12}(t, 0, 0, \nu_0)\mathbb{M}_{22}^{-1}(t, 0, 0, \nu_0)\mathbb{M}_{21}(t, 0, 0, \nu_0). \tag{27}$$

Under these conditions there exist $\tilde{\varepsilon} > 0$, $\tilde{\mu} > 0$, $\tilde{\rho} > 0$, such that for any $0 < \varepsilon \leq \tilde{\varepsilon}$, $0 < \mu \leq \tilde{\mu}$ that satisfy $|\mu^2/\varepsilon - \nu_0| \leq \tilde{\rho}$, the system (19) *is exponentially stable. Furthermore, for these values of the parameters ε and μ, the solutions of the system* (19) *have upper bounds of the form:*

$$\begin{aligned}
&\|\mathbf{X}_1(t,\varepsilon,\mu)\|_1 \leq c_1 e^{-\alpha_s(t-t_0)} (\|\mathbf{X}_1(t_0,\varepsilon,\mu)\|_1 + \varepsilon\|\mathbf{X}_2(t_0,\varepsilon,\mu)\|_2) \\
&\|\mathbf{X}_2(t,\varepsilon,\mu)\|_2 \leq c_2[e^{-\frac{\alpha_f(t-t_0)}{\varepsilon}} \|\mathbf{X}_2(t_0,\varepsilon,\mu)\|_2 + e^{-\alpha_s(t-t_0)} (\|\mathbf{X}_1(t_0,\varepsilon,\mu)\|_1 \\
&\qquad\qquad +\varepsilon\|\mathbf{X}_2(t_0,\varepsilon,\mu)\|_2)]
\end{aligned} \tag{28}$$

for all $t \geq t_0 \geq 0$.

Proof. First, let us remark that the assumption **H1**) together with (20)-(24) guarantee that the assumption (a) from Proposition 1 is fulfilled in the special case of the system (19). The assumptions (b) and (c) from the statement of the Proposition 2 are special forms of the assumptions (b) and (c) from Proposition 1 associated to the system (19) and $\delta_0 = (0, \nu_0)$. So, (28) follows immediately from (10). Thus the proof is complete. □

In the sequel, we shall emphasize some conditions expressed in terms of the coefficients of the system (1) that guarantee that assumptions (b) and (c) from Proposition 2 are fulfilled. For each $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$ we consider the generalized Lyapunov operator $\mathcal{L}_{f\nu_0}(t,i) : \mathcal{S}_{n_2} \to \mathcal{S}_{n_2}$ defined by

$$\mathcal{L}_{f\nu_0}(t,i)[Z] = A_{22}(t,i)Z + ZA_{22}^T(t,i) + \nu_0 \sum_{k=1}^{r} A_{k,22}(t,i)ZA_{k,22}^T(t,i) \tag{29}$$

for all $Z \in \mathcal{S}_{n_2}$. The operator $\mathcal{L}_{f\nu_0}(t,i)$ is the linear operator of Lyapunov type associated to the stochastic linear differential equation of the form

$$dx_2(\tau) = A_{22}(t,i)x_2(\tau)d\tau + \sqrt{\nu_0}\sum_{k=1}^{r} A_{k,22}(t,i)x_2(\tau)dw_k(\tau) \tag{30}$$

where $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$ are parameters.

One knows that for each fixed (t,i), the system (30) is exponentially stable in mean square if and only if the eigenvalues of the operator $\mathcal{L}_{f\nu_0}(t,i)$ are in the half plane $\mathbb{C}_-$.

Now we prove an auxiliary result that will be involved in the proof of the forthcoming proposition.

Lemma 2. *Assume that the assumptions* **H1**) *and* **H2**) *are fulfilled. Under these conditions the following hold:*

(i) *for each* $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$, *the linear operator* $\mathcal{L}_{f\nu_0}(t,i)$ *defines a positive evolution on the linear space* $\mathcal{S}_{n_2}$, *that is,* $e^{\mathcal{L}_{f\nu_0}(t,i)\tau}[Z] \geq 0$, *for all* $\tau \geq 0$ *if* $Z \geq 0$;

(ii) *if the eigenvalues of the linear operators* $\mathcal{L}_{f\nu_0}(t,i)$ *are placed in the half plane of the form* $\mathbb{C}_\lambda$, *with* $\lambda > 0$ *not depending upon* $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$, *then, for any* $\tilde{\lambda} \in (0,\lambda)$ *there exists* $\tilde{c} > 0$ *not depending upon* t, i, τ *such that* $\|e^{\mathcal{L}_{f\nu_0}(t,i)\tau}\| \leq \tilde{c}e^{-\tilde{\lambda}\tau}$, *for all* $\tau \geq 0$, $\|\cdot\|$ *being the operator norm induced by the spectral norm* $|\cdot|$ *on* $\mathcal{S}_{n_2}$.

Proof. The assertion (i) is a straightforward consequence of the Remark 2.6.2 from [7].

To prove (ii) let us remark that

$$\|e^{\mathcal{L}_{f\nu_0}(t,i)\tau}\| = |e^{\mathcal{L}_{f\nu_0}(t,i)\tau}[I_{n_2}]|,$$

where I_{n_2} is the identity matrix of size $n_2 \times n_2$. To obtain this equality we have used the Corollary 2.1.7 (i) and Theorem 2.1.10 from [7] in the special case of the Banach space $(\mathcal{S}_{n_2}, |\cdot|)$. We set $Z(\tau) = e^{\mathcal{L}_{f\nu_0}(t,i)\tau}[I_{n_2}]$. So, $Z(\cdot)$ is the solution of the problem with given initial values:

$$\frac{d}{d\tau}Z(\tau) = \mathcal{L}_{f\nu_0}(t,i)[Z(\tau)], \quad Z(0) = I_{n_2}. \tag{31}$$

Using the techniques of H-representation, introduced in [23] we obtain that $Z(\tau) = \varphi^{-1}(\xi(\tau))$ where $\varphi : \mathcal{S}_{n_2} \to \mathbb{R}^{\frac{n_2(n_2+1)}{2}}$ is the isomorphism introduced in [23] and $\xi(\tau)$ is the solution of the following problem with given initial values:

$$\frac{d}{d\tau}\xi(\tau) = \mathbf{\Theta}(t,i)\xi(\tau), \quad \xi(0) = \varphi(I_{n_2}), \tag{32}$$

where $\mathbf{\Theta}(t,i)$ is the matrix associated to the linear operator $\mathcal{L}_{f\nu_0}(t,i)$ via (9) from [23]. Invoking the boundedness of the functions $t \to A_{22}(t,i)$ and $t \to A_{k,22}(t,i)$ we may deduce that there exists $\check{\gamma} > 0$ not depending upon $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$ such that $|\mathbf{\Theta}(t,i)| \leq \check{\gamma}$.

On the other hand, from Lemma 3.1 (ii) from [23] applied in the case of linear operator defined in (29), we infer that the spectrum of the operator $\mathcal{L}_{f\nu_0}(t,i)$ coincides to the spectrum of the matrix $\mathbf{\Theta}(t,i)$.
We consider $\tilde{\lambda} \in (0,\lambda)$. Applying Proposition 3, Chapter 1 from [1] in the case of the matrix $\mathbf{\Theta}(t,i)$ we deduce that there exists $\tilde{c}_1 > 0$ not depending

upon (t, i) so that $|e^{\mathbf{\Theta}(t,i)\tau}| \leq \tilde{c}_1 e^{-\tilde{\lambda}\tau}$, for all $\tau \geq 0$. This allows us to deduce that the solutions of the problem with given initial values satisfy

$$|\xi(\tau)| \leq \tilde{c}_1 e^{-\tilde{\lambda}\tau} |\varphi(I_{n_2})|.$$

In this way we may conclude that

$$\|e^{\mathcal{L}_{f\nu_0}(t,i)\tau}\| = |Z(\tau)| = |\varphi^{-1}(\xi(\tau))| \leq \tilde{c} e^{-\tilde{\lambda}\tau}$$

where $\tilde{c} = \|\varphi^{-1}\| \cdot |\varphi(I_{n_2})| \tilde{c}_1$. Thus the proof is complete. □

The next result provides a condition which guarantee that the assumption (b) from Proposition 2 is fulfilled.

Proposition 3. *Assume:*

(a) **H1**) *and* **H2**) *are fulfilled;*

(b) *there exists* $\lambda > 0$ *not depending upon* (t, i) *such that the eigenvalues of the linear operators* $\mathcal{L}_{f\nu_0}(t, i)$ *are located in the half plane* $\mathbb{C}_\lambda$ *for all* $(t, i) \in \mathbb{R}_+ \times \mathfrak{N}$.

Under these conditions there exists $\alpha_f > 0$ *not depending upon* t *such that the eigenvalues of the linear operator* $\mathbb{M}_{22}(t, 0, 0, \nu_0)$ *are in the half plane* $\mathbb{C}_{2\alpha}$.

Proof. The conclusion one obtains showing that there exist $\hat{c} \geq 1$ and $\alpha > 0$ not depending upon $t \in \mathbb{R}_+$ such that

$$\|e^{\mathbb{M}_{22}(t,0,0,\nu_0)\tau}\| \leq \hat{c} e^{-\alpha\tau},$$

for all $\tau \geq 0$, $t \in \mathbb{R}_+$. Here, $\|\cdot\|$ is the operator norm induced by the norm $\|\cdot\|_2$ on the Banach space $\mathcal{X}_2$. Employing (24) and (29) we obtain that the linear differential equation on $\mathcal{X}_2$:

$$\frac{d}{d\tau}\mathbf{X}_2(\tau) = \mathbb{M}_{22}(t, 0, 0, \nu_0)\mathbf{X}_2(\tau)$$

with t as a parameter, has the partition:

$$\begin{aligned}
&\frac{d}{d\tau}X_{12}(\tau, i) = X_{12}(\tau, i)A_{22}^T(t, i) \\
&\frac{d}{d\tau}X_{22}(\tau, i) = \mathcal{L}_{f\nu_0}(t, i)[X_{22}(\tau, i)] + A_{21}(t, i)X_{12}(\tau, i) + X_{12}^T(\tau, i)A_{21}^T(t, i).
\end{aligned}$$

We deduce that for each $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$ we have the representation:

$$X_{12}(\tau,i) = X_{12}(0,i)e^{A_{22}^T(t,i)\tau}, \ \ \forall\, \tau \geq 0,\ X_{12}(0,i) \in \mathbb{R}^{n_1 \times n_2} \tag{33}$$

$$\begin{aligned} X_{22}(\tau,i) = e^{\mathcal{L}_{f\nu_0}(t,i)\tau}[X_{22}(0,i)] + \int_0^\tau e^{\mathcal{L}_{f\nu_0}(t,i)(\tau-\sigma)}[A_{21}(t,i)X_{12}(\tau,i) \\ + X_{12}^T(\tau,i)A_{21}^T(t,i)]d\sigma \end{aligned} \tag{34}$$

for all $\tau \geq 0$, $X_{22}(0,i) \in \mathcal{S}_{n_2}$. Based on Lemma 2 we infer that under the considered assumptions we have:

$$\|e^{\mathcal{L}_{f\nu_0}(t,i)\tau}\| \leq \tilde{c}e^{-\tilde{\lambda}\tau}, \tag{35}$$

for all $\tau \geq 0$, where $\tilde{c} \geq 1$, $\tilde{\lambda} > 0$ not depending upon $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$. Setting $\nu_0 = 0$ in (29) we obtain

$$\mathcal{L}_{f0}(t,i)[Z] = A_{22}(t,i)Z + ZA_{22}^T(t,i). \tag{36}$$

We have $\mathcal{L}_{f0}(t,i) \leq \mathcal{L}_{f\nu_0}(t,i)$ that yields

$$0 \leq e^{\mathcal{L}_{f0}(t,i)\tau}[Z] \leq e^{\mathcal{L}_{f\nu_0}(t,i)\tau}[Z] \ \text{ if } \ Z \geq 0.$$

Applying Corollary 2.1.11 from [7] in the special case of this positive operator we obtain

$$\|e^{\mathcal{L}_{f0}(t,i)\tau}\| \leq \|e^{\mathcal{L}_{f\nu_0}(t,i)\tau}\|, \tag{37}$$

for all $\tau \geq 0$, $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$. Here and in (35), $\|\cdot\|$ is the operator norm induced by the spectral norm $|\cdot|$ on $\mathcal{S}_{n_2}$. From (35) and (37) we get

$$\|e^{\mathcal{L}_{f0}(t,i)\tau}\| \leq \tilde{c}e^{-\tilde{\lambda}\tau} \tag{38}$$

for all $\tau \geq 0$, $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$. On the other hand, employing Theorem 2.1.10 from [7] in the special case of the operator $\mathcal{L}_{f0}(t,i)$ we have

$$\|e^{\mathcal{L}_{f0}(t,i)\tau}\| = |e^{\mathcal{L}_{f0}(t,i)\tau}[I_{n_2}]|.$$

According to (36) we obtain that

$$e^{\mathcal{L}_{f0}(t,i)\tau}[I_{n_2}] = e^{A_{22}(t,i)\tau}e^{A_{22}^T(t,i)\tau}.$$

Hence, (38) becomes

$$|e^{A_{22}(t,i)\tau}e^{A_{22}^T(t,i)\tau}| \leq \tilde{c}e^{-\tilde{\lambda}\tau}$$

that yields

$$|e^{A_{22}^T(t,i)\tau}| \leq \sqrt{\tilde{c}}e^{-\frac{\tilde{\lambda}}{2}\tau} \tag{39}$$

for all $\tau \geq 0$, $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$. The proof may continue as in the time invariant case employing (33)-(35) and (39), because the upper bounds from (35) and (39) are uniform with respect to $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$. For details see for example Lemma 1 from [9]. $\square$

Assuming that $A_{22}(t,i)$ are invertible, we introduce the notations:

$$\begin{aligned} A_s(t,i) &= A_{11}(t,i) - A_{12}(t,i)A_{22}^{-1}(t,i)A_{21}(t,i) \\ A_{kl,s}(t,i) &= A_{k,l1}(t,i) - A_{k,l2}(t,i)A_{22}^{-1}(t,i)A_{21}(t,i) \end{aligned} \tag{40}$$

$l = 1, 2$, $1 \leq k \leq r$, $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$. Assuming also that for each $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$ the linear operator $\mathcal{L}_{f\nu_0}(t,i)$ defined by (29) is invertible, we introduce the following linear operator $\mathcal{L}_{s\nu_0}(t) : \mathcal{S}_{n_1}^N \to \mathcal{S}_{n_1}^N$ defined by

$$\mathcal{L}_{s\nu_0}(t)[\mathbf{X}_1] = (\mathcal{L}_{s\nu_0}(t)[\mathbf{X}_1](1), \ldots, \mathcal{L}_{s\nu_0}(t)[\mathbf{X}_1](N))$$

with

$$\begin{aligned} &\mathcal{L}_{s\nu_0}(t)[\mathbf{X}_1](i) \\ &\quad = A_s(t,i)X_1(i) + X_1(i)A_s^T(t,i) + \sum_{k=1}^{r} A_{k1,s}(t,i)X_1(i)A_{k1,s}^T(t,i) \\ &\quad - \nu_0 \sum_{k=1}^{r}\sum_{l=1}^{r} A_{k,12}(t,i) \\ &\quad \times \mathcal{L}_{f\nu_0}^{-1}(t,i)[A_{l2,s}(t,i)X_1(i)A_{l2,s}^T(t,i)]A_{k,12}^T(t,i) + \sum_{j=1}^{N} q_{ji}X_1(j \end{aligned} \tag{41}$$

for all $\mathbf{X}_1 = (X_1(1), \ldots, X_1(N)) \in \mathcal{S}_{n_1}^N$.

The next result provides an explicit formula of the operator $\mathbb{M}_s(t)$ defined in (27).

Lemma 3. *If for any $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$, the matrices $A_{22}(t,i)$ and the linear operator $\mathcal{L}_{f\nu_0}(t,i)$ are invertible, then we have $\mathbb{M}_s(t) = \mathcal{L}_{s\nu_0}(t)$, for all $t \in \mathbb{R}_+$.*

The proof is done by direct calculation involving (20)-(24) written for $\varepsilon = 0$, $\mu = 0$, $\nu = \nu_0$.

One of the main results of this paper is:

Theorem 1. *Assume:*

(a) *the assumption* **H1**) *and the assumption* **H2**) *with* $\nu_0 > 0$ *are fulfilled;*

(b) *for any* $(t,i) \in \mathbb{R}_+ \times \mathfrak{N}$ *the eigenvalues of the linear operators* $\mathcal{L}_{f\nu_0}(t,i)$ *are located into the half plane of the form* $\{z \in \mathbb{C} | \mathrm{Re}\ z \le -2\alpha_f\}$ *for an* $\alpha_f > 0$ *not depending upon* (t,i);

(c) *the linear evolution operator* $\mathbf{T}_s(t,t_0)$ *defined by the linear differential equation on* $\mathcal{S}_{n_1}^N$

$$\dot{\mathbf{X}}_1(t) = \mathcal{L}_{s\nu_0}(t)[\mathbf{X}_1(t)]$$

has the behaviour of the form

$$\|\mathbf{T}_s(t,t_0)\| \le \beta_s e^{-2\alpha_s(t-t_0)}$$

for all $t \ge t_0 \ge 0$, $\beta_s \ge 1$, $\alpha_s > 0$, *being constants not depending upon* t *and* t_0.

Under these conditions there exist $\varepsilon^* > 0$, $\mu^* > 0$, $\rho^* > 0$, *with the properties that the full system of SDEs* (1) *is ESMS for arbitrary* $0 < \varepsilon \le \varepsilon^*$, $0 < \mu \le \mu^*$, *such that* $|\mu^2/\varepsilon - \nu_0| \le \rho^*$. *Moreover, if*

$$\begin{bmatrix} \phi_{11}(t,t_0;\varepsilon,\mu) & \phi_{12}(t,t_0;\varepsilon,\mu) \\ \phi_{21}(t,t_0;\varepsilon,\mu) & \phi_{22}(t,t_0;\varepsilon,\mu) \end{bmatrix}$$

is the partition of the fundamental matrix solution $\phi(t,t_0;\varepsilon,\mu)$ *of the system* (1) *compatible with the partition of the coefficients of the system* (4) *we have*

$$\begin{aligned} &\mathbb{E}[|\phi_{j1}(t,t_0;\varepsilon,\mu)|^2 | \eta_{t_0} = i] \le c_{j1} e^{-\alpha_1(t-t_0)},\ j = 1,2, \\ &\mathbb{E}[|\phi_{12}(t,t_0;\varepsilon,\mu)|^2 | \eta_{t_0} = i] \le c_{12} \varepsilon e^{-\alpha_1(t-t_0)}, \\ &\mathbb{E}[|\phi_{22}(t,t_0;\varepsilon,\mu)|^2 | \eta_{t_0} = i] \le c_{22}(e^{-\frac{\alpha_2(t-t_0)}{\varepsilon}} + \varepsilon e^{-\alpha_1(t-t_0)}) \end{aligned} \tag{42}$$

for all $t \ge t_0 \ge 0$, $i \in \mathfrak{N}$, *where* $c_{jl} \ge 1$, $\alpha_1 \in (0,\alpha_s)$ *and* $\alpha_2 \in (0,\alpha_f)$ *do not depend upon* t, t_0, ε *and* μ.

Proof. Based on Proposition 3 and Lemma 3 we may remark that, if the assumptions in the statement are fulfilled then the assumptions of the Proposition 2 are also satisfied. Hence, there exists $\varepsilon^* > 0$, $\mu^* > 0$, with the property that if $0 < \varepsilon \le \varepsilon^*$, $0 < \mu \le \mu^*$ and $|\mu^2/\varepsilon - \nu_0| \le \rho^*$, then the solutions of the system (19) or, equivalently, the solutions of the system (15) satisfy inequalities of the form (28). To simplify the notations, in the rest of the proof we shall omit the dependence with respect to ε and μ of the solutions of equations (15) as well as of the block components of the fundamental matrix solution $\phi(t, t_0; \varepsilon, \mu)$ of the system (1). Let

$$\mathbf{X}(t, t_0, \mathbf{H}) = (X(t, t_0, \mathbf{H}, 1), X(t, t_0, \mathbf{H}, 2), \dots, X(t, t_0, \mathbf{H}, N))$$

be the solution of the system (15) satisfying the initial condition $\mathbf{X}(t_0, t_0, \mathbf{H}) = \mathbf{H}$, where $\mathbf{H}$ is arbitrary in $\mathcal{S}_n^N$.

The j-th component of the solution $\mathbf{X}(t, t_0, \mathbf{H})$ has the representation

$$X(t, t_0, \mathbf{H}, j) = \mathbf{T}(t, t_0)[\mathbf{H}](j), \ \ 1 \le j \le N,$$

$\mathbf{T}(t, t_0)$ being the linear evolution operator defined by the linear differential equation (15).

Further, employing the representation formula of a linear evolution operator defined by a Lyapunov type linear differential equation associated to linear SDEs (see for example formula (3.8) from Remark 3.1.3 in [7]) we obtain

$$X(t, t_0, \mathbf{H}, j) = \sum_{i=1}^{N} \mathbb{E}[\phi(t, t_0) H(i) \phi^T(t, t_0) \chi_{\{\eta_t = j\}} | \eta_{t_0} = i]. \tag{43}$$

Let $i_0 \in \mathfrak{N}$ be arbitrary but fixed and $\mathbf{H}_{i_0} \in \mathcal{S}_n^N$ be defined by $\mathbf{H}_{i_0} = (H_{i_0}(1), \dots, H_{i_0}(N))$, where

$$H_{i_0}(i) = \begin{cases} 0, & \text{if } i \ne i_0 \\ x_0 x_0^T, & \text{if } i = i_0, \end{cases}$$

with $x_0 = (x_{10}^T \ \ x_{20}^T)^T \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is arbitrary but fixed. The equalities (43) written for $\mathbf{H}$ replaced by $\mathbf{H}_{i_0}$ yield:

$$\mathbb{E}[\phi(t, t_0) x_0 (\phi(t, t_0) x_0)^T | \eta_{t_0} = i_0] = \sum_{j=1}^{N} X(t, t_0, \mathbf{H}_{i_0}, j). \tag{44}$$

Let

$$\begin{bmatrix} X_{11}(t, t_0, \mathbf{H}_{i_0}, j) & X_{12}(t, t_0, \mathbf{H}_{i_0}, j) \\ X_{12}^T(t, t_0, \mathbf{H}_{i_0}, j) & X_{22}(t, t_0, \mathbf{H}_{i_0}, j) \end{bmatrix}$$

be the partition of the matrix $X(t, t_0, \mathbf{H}_{i_0}, j)$ compatible with the partition of the coefficients of (15) given in (3). We set

$$\mathbf{X}_1^{i_0}(t) = (X_{11}(t, t_0, \mathbf{H}_{i_0}, 1), \dots, X_{11}(t, t_0, \mathbf{H}_{i_0}, N))$$

and $\mathbf{X}_2^{i_0}(t) = [(X_{12}(t,t_0,\mathbf{H}_{i_0},1), X_{22}(t,t_0,\mathbf{H}_{i_0},1)), \ \dots\ ,$ $(X_{12}(t,t_0,\mathbf{H}_{i_0},N), X_{22}(t,t_0,\mathbf{H}_{i_0},N))]$.

Based on the partition given in (16) of the equation (15) satisfied by $\mathbf{X}(t,t_0,\mathbf{H}_{i_0})$ we deduce that $t \to \left(\mathbf{X}_1^{i_0}(t), \mathbf{X}_2^{i_0}(t)\right)$ is a solution of the system (19).

On the other hand, (17) and (18) respectively, allow us to obtain that

$$\|\mathbf{X}_1^{i_0}(t_0)\|_1 = |x_{10}|^2 \ \text{ and } \ \|\mathbf{X}_2^{i_0}(t_0)\|_2 = \max\{|x_{10}x_{20}^T|, |x_{20}|^2\}. \tag{45}$$

Taking $x_{20} = 0$ in (44) we obtain

$$\begin{aligned} &\mathbb{E}[\phi_{11}(t,t_0)x_{10}(\phi_{11}(t,t_0)x_{10})^T|\eta_{t_0} = i_0] \\ &\quad = \sum_{j=1}^{N} X_{11}(t,t_0,\check{\mathbf{H}}_{i_0},j) \le N\|\check{\mathbf{X}}_1^{i_0}(t)\|_1 I_{n_1} \end{aligned} \tag{46a}$$

$$\begin{aligned} &\mathbb{E}[\phi_{21}(t,t_0)x_{10}(\phi_{21}(t,t_0)x_{10})^T|\eta_{t_0} = i_0] \\ &\quad = \sum_{j=1}^{N} X_{22}(t,t_0,\check{\mathbf{H}}_{i_0},j) \le N\|\check{\mathbf{X}}_2^{i_0}(t)\|_2 I_{n_2} \end{aligned} \tag{46b}$$

where $\check{\mathbf{H}}_{i_0}$ is the value of $\mathbf{H}_{i_0}$ in the case of $x_0 = (x_{10}^T\ 0^T)^T$ and $(\check{\mathbf{X}}_1^{i_0}(t), \check{\mathbf{X}}_2^{i_0}(t))$ stands for $(\mathbf{X}_1^{i_0}(t), \mathbf{X}_2^{i_0}(t))$ when $\mathbf{H}_{i_0}$ is replaced by $\check{\mathbf{H}}_{i_0}$.

On the other hand if $x_{10} = 0$, then (44) yields

$$\begin{aligned} &\mathbb{E}[\phi_{12}(t,t_0)x_{20}(\phi_{12}(t,t_0)x_{20})^T|\eta_{t_0} = i_0] \\ &\quad = \sum_{j=1}^{N} X_{11}(t,t_0,\hat{\mathbf{H}}_{i_0},j) \le N\|\hat{\mathbf{X}}_1^{i_0}(t)\|_1 I_{n_1} \end{aligned} \tag{47a}$$

$$\begin{aligned} &\mathbb{E}[\phi_{22}(t,t_0)x_{20}(\phi_{22}(t,t_0)x_{20})^T|\eta_{t_0} = i_0] \\ &\quad = \sum_{j=1}^{N} X_{22}(t,t_0,\hat{\mathbf{H}}_{i_0},j) leq N\|\hat{\mathbf{X}}_2^{i_0}(t)\|_2 I_{n_2} \end{aligned} \tag{47b}$$

where $\hat{\mathbf{H}}_{i_0}$ is the value of $\mathbf{H}_{i_0}$ when $x_0 = (0^T\ x_{20}^T)^T$ and $(\hat{\mathbf{X}}_1^{i_0}(t), \hat{\mathbf{X}}_2^{i_0}(t))$ stands for $(\mathbf{X}_1^{i_0}(t), \mathbf{X}_2^{i_0}(t))$ when $\mathbf{H}_{i_0}$ is replaced by $\hat{\mathbf{H}}_{i_0}$.

Employing (28), (45), (46) we get

$$\mathbb{E}[|\phi_{j1}(t,t_0)x_{10}|^2|\eta_{t_0} = i_0] \le n_j c_j N e^{-\alpha_1(t-t_0)}|x_{10}|^2$$

for all $t \ge t_0 \ge 0$, $j = 1, 2$. Thus, we obtained the first two inequalities from (42). The other inequalities from (42) are obtained combining (28) with (45) and (47). Thus the proof is complete. $\square$

Remark 2. (a) *The assumption* (*b*) *of Theorem 1 could be reformulated in terms of uniform exponential stability in mean square with respect to* $(t, i) \in \mathbb{R}_+ \times \mathfrak{N}$ *of the system with frozen coefficients* (30). *The assumption* (*c*) *of the same theorem cannot be expressed in terms of exponential stability in mean square for a system of SDE, because if* $\nu_0 > 0$ *the linear operator* $\mathcal{L}_{s\nu_0}(t)$ *defined in* (41) *cannot be interpreted into an obvious way as an Lyapunov type operator associated to a system of SDEs.*

(b) *Following the same line of the proof as in the time invariant case* (*see for example Proposition 2 from* [9]) *one may show that the linear operator* $\mathcal{L}_{s\nu_0}(t)$ *defines a positive evolution on the space* $\mathcal{S}_{n_1}^N$, *i.e.* $\mathbf{T}_s(t, t_0)[\mathbf{X}_1] \geq 0$, *for all* $t \geq t_0$ *if* $\mathbf{X}_1 = (X_1(1), \ldots, X_1(N))$ *is such that* $X_1(i) \geq 0$, *for all* $i \in \mathfrak{N}$. *This fact could be exploited to obtain sufficient conditions which guarantee that the assumption* (c) *of Theorem 1 is fulfilled.*

In the sequel we focus our attention on the case when the assumption **H2**) is fulfilled with $\nu_0 = 0$. In this case, the linear operator introduced in (41) reduces to

$$\mathcal{L}_{s0}(t)[\mathbf{X}_1] = (\mathcal{L}_{s0}(t)[\mathbf{X}_1](1), \ldots, \mathcal{L}_{s0}(t)[\mathbf{X}_1](N)),$$

where

$$\begin{aligned}\mathcal{L}_{s0}(t)[\mathbf{X}_1](i) &= A_s(t,i)X_1(i) + X_1(i)A_s^T(t,i) \\ &\quad + \sum_{k=1}^{r} A_{k1,s}(t,i)X_1(i)A_{k1,s}^T(t,i) + \sum_{j=1}^{N} q_{ji}X_1(j) \end{aligned} \tag{48}$$

for all $\mathbf{X}_1 \in \mathcal{S}_{n_1}^N$. $A_s(t,i)$ and $A_{k1,s}(t,i)$ being defined in (40). One sees that the linear operator $\mathcal{L}_{s0}(t)$ may be viewed as the Lyapunov type operator associated to the following system of SDEs.

$$dx_1(t) = A_s(t,\eta_t)x_1(t)dt + \sum_{k=1}^{r} A_{k1,s}(t,\eta_t)x_1(t)dw_k(t). \tag{49}$$

The system (49) can be obtained from the system (1) setting formal $\varepsilon = 0$, $\mu = 0$. This is why, it is named the reduced subsytem, or the slow subsystem associated to (1).

The second main result of this work is

Theorem 2. *Assume:*

(a) *the assumption* **H1**) *and the assumption* **H2**) *with* $\nu_0 = 0$ *are fulfilled.*

(b) *There exists* $\alpha_f > 0$ *not depending upon* (t, i) *such that the eigenvalues of the matrix* $A_{22}(t, i)$ *are in the half plane* $\{z \in \mathbb{C} | \mathrm{Re}\ z \leq -\alpha_f\}$ *for all* $(t, i) \in \mathbb{R}_+ \times \mathfrak{N}$.

(c) *The reduced subsystem* (49) *associated to* (1) *is exponentially stable in mean square.*

Under these conditions there exist $\tilde{\tilde{\varepsilon}} > 0$, $\tilde{\tilde{\mu}} > 0$, $\tilde{\tilde{\rho}} > 0$ *with the property that the full system (1) is ESMS for arbitrary* $\varepsilon \in (0, \tilde{\tilde{\varepsilon}}]$, $\mu \in (0, \tilde{\tilde{\mu}}]$ *which satisfies* $\mu^2/\varepsilon \in (0, \tilde{\tilde{\rho}}]$. *For these values of the small parameters* $\varepsilon > 0$, $\mu > 0$, *the block components of the fundamental matrix solution* $\phi(t, t_0; \varepsilon, \mu)$ *of the system* (1) *have upper bounds of the type* (42).

Proof. (Hint) As in the case of Theorem 1 one shows that the assumptions from the statement guarantee that the assumptions of Proposition 2 are fullfilled in the special case of the system (19) when $\nu_0 = 0$. The details are omitted. Here, we only recall that in this special case, the linear operator (29) reduces to $\mathcal{L}_{f0}(t, i)$ given in (36). One checks that if the eigenvalues of the matrices $A_{22}(t, i)$ are in the half plane of type $\{z \in \mathbb{C} | \mathrm{Re}\ z \leq -\alpha_f\}$ then the eigenvalues of the linear operator defined in (36) are in the half plane $\{z \in \mathbb{C} | \mathrm{Re}\ z \leq -2\alpha_f\}$. □

5 Conclusion

In this paper, the stability problem for a class of large-scale singularly perturbed linear stochastic systems with state-multiplicative white noise and Markovian jumping parameters has been investigated. Based on the linear evolution operator, an exponential stability in mean square with the decay rates for the slow and the fast subsystems has been evaluated by using the stochastic Lyapunov differential equation. Thus, the present results are the extension to the time varying case of [9].

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