DERIVED CONES TO REACHABLE SETS OF A SECOND-ORDER EVOLUTION INCLUSION *

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Dedicated to Professor Mihail Megan
on the occasion of his 70th anniversary

Abstract

We consider a class of second-order evolution inclusions and we prove that the reachable set of a certain second-order variational inclusion is a derived cone in the sense of Hestenes to the reachable set of the initial differential inclusion. This result allows to obtain a sufficient condition for local controllability along a reference trajectory.

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1 Introduction

The concept of derived cone to an arbitrary subset of a normed space has been introduced by M.Hestenes in [8] and successfully used to obtain necessary optimality conditions in Control Theory. Afterwards, this concept has been largely ignored in favor of other concepts of tangents cones, that may intrinsically be associated to a point of a given set: the cone of interior directions, the contingent, the quasitangent and, above all, Clarke's tangent cone

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(e.g., [1]). Mirică ([12,13]) obtained "an intersection property" of derived cones that allowed a conceptually simple proof and significant extensions of the maximum principle in optimal control; moreover, other properties of derived cones may be used to obtain controllability and other results in the qualitative theory of control systems.

In this paper we are concerned with the following problem

$$x'' \in A(t)x + F(t,x), \quad x(0) \in X_0, \quad x'(0) \in X_1, \qquad (1.1)$$

where $F : [0,T] \times X \to \mathcal{P}(X)$ is a set-valued map, X is a separable Banach space, $X_0, X_1 \subset X$ and $\{A(t)\}_{t \geq 0}$ is a family of linear closed operators from X into X that generates an evolution system of operators $\{\mathcal{U}(t,s)\}_{t,s \in [0,T]}$.

The general framework of evolution operators $\{A(t)\}_{t \geq 0}$ that define problem (1.1) has been developed by Kozak ([11]) and improved by Henriquez ([9]). In several recent papers ([2-5], [9,10]) existence results and qualitative properties of solutions for problem (1.1) have been obtained by using fixed point techniques.

Our aim is to prove that the reachable set of a certain second-order variational inclusion is a derived cone in the sense of Hestenes to the reachable set of the problem (1.1). In order to obtain the continuity property in the definition of a derived cone we shall use a continuous version of Filippov's theorem for mild solutions of differential inclusions (1.1) obtained in [7]. As an application, when X is finite dimensional, we obtain a sufficient condition for local controllability along a reference trajectory. The results in this paper may be interpreted as an extension of the results in our previous paper [6] obtained for second-order differential inclusions defined by cosine family of operators to the more general problem (1.1).

The paper is organized as follows: in Section 2 we present the notations and the preliminary results to be used in the sequel and in Section 3 we provide our main results.

2   Preliminaries

Since the reachable set to a control system is, generally, neither a differentiable manifold, nor a convex set, its infinitesimal properties may be characterized only by tangent cones in a generalized sense, extending the classical concepts of tangent cones in Differential Geometry and Convex Analysis, respectively.

From the rather large number of "convex approximations", "tents", "regular tangents cones", etc. in the literature, we choose the concept of derived

cone introduced by M. Hestenes in [8].

Let $(X, ||.||)$ be a normed space.

Definition 2.1 ([8]). A subset $M \subset X$ is said to be a *derived set to* $E \subset X$ at $x \in E$ if for any finite subset $\{v_1, ..., v_k\} \subset M$, there exist $s_0 > 0$ and a continuous mapping $a(.) : [0, s_0]^k \to E$ such that $a(0) = x$ and $a(.)$ is (conically) differentiable at $s = 0$ with the derivative $\mathrm{col}[v_1, ..., v_k]$ in the sense that

$$\lim_{\mathbf{R}_+^k \ni \theta \to 0} \frac{||a(\theta) - a(0) - \sum_{i=1}^{k} \theta_i v_i||}{||\theta||} = 0.$$

We shall write in this case that the derivative of $a(.)$ at $s = 0$ is given by

$$\mathrm{D}a(0)\theta = \sum_{i=1}^{k} \theta_j v_j \quad \forall \theta = (\theta_1, ..., \theta_k) \in \mathbf{R}_+^k := [0, \infty)^k.$$

A subset $C \subset X$ is said to be a *derived cone* of E at x if it is a derived set and also a convex cone.

For the basic properties of derived sets and cones we refer to M. Hestenes [8]; we recall that if M is a derived set then $M \bigcup \{0\}$ as well as the convex cone generated by M, defined by

$$\mathrm{cco}(M) = \{\sum_{i=1}^{k} \lambda_j v_j; \quad \lambda_j \geq 0, \, k \in N, \, v_j \in M, \, j = 1, ..., k\}$$

is also a derived set, hence a derived cone.

The fact that the derived cone is a proper generalization of the classical concepts in Differential Geometry and Convex Analysis is illustrated by the following results ([8]): if $E \subset \mathbf{R}^n$ is a differentiable manifold and $T_x E$ is the tangent space in the sense of Differential Geometry to E at x

$$T_x E = \{v \in \mathbf{R}^n; \, \exists \, c : (-s, s) \to X, \, \text{of class } C^1, c(0) = x, c'(0) = v\},$$

then $T_x E$ is a derived cone; also, if $E \subset \mathbf{R}^n$ is a convex subset then the tangent cone in the sense of Convex Analysis defined by

$$TC_x E = \mathrm{cl}\{t(y - x); \quad t \geq 0, \, y \in E\}$$

is also a derived cone. Since any convex subcone of a derived cone is also a derived cone, such an object may not be uniquely associated to a point $x \in E$; moreover, simple examples show that even a maximal with respect

to set-inclusion derived cone may not be uniquely defined: if the set $E \subset \mathbf{R}^2$ is defined by

$$E = C_1 \bigcup C_2, \; C_1 = \{(x,x); x \geq 0\}, \; C_2 = \{(x,-x), x \leq 0\},$$

then C_1 and C_2 are both maximal derived cones of E at the point $(0,0) \in E$.

On the other hand, the up-to-date experience in Nonsmooth Analysis shows that for some problems, the use of one of the intrinsic tangent cones may be preferable. From the multitude of the intrinsic tangent cones in the literature (e.g. [1]), the contingent, the quasitangent (intermediate) and Clarke's tangent cones, defined, respectively, by

$$K_x E = \{v \in X; \quad \exists s_m \to 0+, \quad \exists x_m \to x, \; x_m \in E : \tfrac{x_m - x}{s_m} \to v\},$$
$$Q_x E = \{v \in X; \quad \forall s_m \to 0+, \quad \exists x_m \to x, \; x_m \in E : \tfrac{x_m - x}{s_m} \to v\},$$
$$C_x E = \{v \in X; \quad \forall (x_m, s_m) \to (x, 0+), \quad x_m \in E, \exists y_m \in E : \tfrac{y_m - x_m}{s_m} \to v\}$$

seem to be among the most often used in the study of different problems involving nonsmooth sets and mappings.

An outstanding property of derived cone, obtained by Hestenes ([8], Theorem 4.7.4) is stated in the next lemma.

Lemma 2.2. *Let* $X = \mathbf{R}^n$. *Then* $x \in \text{int}(E)$ *if and only if* $C = \mathbf{R}^n$ *is a derived cone at* $x \in E$ *to* E.

Corresponding to each type of tangent cone, say $\tau_x E$ one may introduce (e.g. [1]) a *set-valued directional derivative* of a multifunction $G(.) : E \subset X \to \mathcal{P}(X)$ (in particular of a single-valued mapping) at a point $(x,y) \in \text{Graph}(G)$ as follows

$$\tau_y G(x;v) = \{w \in X; \quad (v,w) \in \tau_{(x,y)} \text{Graph}(G)\}, \quad v \in \tau_x E.$$

We recall that a set-valued map, $A(.) : X \to \mathcal{P}(X)$ is said to be a *convex* (respectively, closed convex) *process* if $\text{Graph}(A(.)) \subset X \times X$ is a convex (respectively, closed convex) cone. For the basic properties of convex processes we refer to [1], but we shall use here only the above definition.

Let denote by I the interval $[0, T]$ and let X be a real separable Banach space with the norm $||.||$ and with the corresponding metric $d(.,.)$. Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$\mathrm{d}_H(A,B) = \max\{\mathrm{d}_H^*(A,B), \mathrm{d}_H^*(B,A)\},$$

where $d_H^*(A, B) = \sup\{d(a, B); a \in A\}$ and $d(a, B) = \inf_{y \in B} d(a, y)$.

Let (Z, d) be a metric space and consider a set valued map T on Z with nonempty closed values in Z. T is said to be λ-Lipschitz if there exists $\lambda > 0$ such that

$$d_H(T(x), T(y)) \le \lambda d(x, y) \quad \forall x, y \in Z.$$

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $||x(.)||_C = \sup_{t \in I} ||x(t)||$, by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $||x(.)||_1 = \int_I ||x(t)|| dt$ and by $B(X)$ the Banach space of linear bounded operators on X.

In what follows $\{A(t)\}_{t \ge 0}$ is a family of linear closed operators from X into X that genearates an evolution system of operators $\{\mathcal{U}(t, s)\}_{t, s \in I}$. By hypothesis the domain of $A(t)$, $D(A(t))$ is dense in X and is independent of t.

Definition 2.3. ([9,11]) A family of bounded linear operators $\mathcal{U}(t, s)$: $X \to X$, $(t, s) \in \Delta := \{(t, s) \in I \times I; s \le t\}$ is called an evolution operator of the equation

$$x''(t) = A(t)x(t) \tag{2.1}$$

if

i) For any $x \in X$, the map $(t, s) \to \mathcal{U}(t, s)x$ is continuously differentiable and

    a) $\mathcal{U}(t, t) = 0$, $t \in I$.

    b) If $t \in I, x \in X$ then $\frac{\partial}{\partial t}\mathcal{U}(t, s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}\mathcal{U}(t, s)x|_{t=s} = -x$.

ii) If $(t, s) \in \Delta$, then $\frac{\partial}{\partial s}\mathcal{U}(t, s)x \in D(A(t))$, the map $(t, s) \to \mathcal{U}(t, s)x$ is of class C^2 and

    a) $\frac{\partial^2}{\partial t^2}\mathcal{U}(t, s)x \equiv A(t)\mathcal{U}(t, s)x$.

    b) $\frac{\partial^2}{\partial s^2}\mathcal{U}(t, s)x \equiv \mathcal{U}(t, s)A(t)x$.

    c) $\frac{\partial^2}{\partial s \partial t}\mathcal{U}(t, s)x|_{t=s} = 0$.

iii) If $(t, s) \in \Delta$, then there exist $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t, s)x$, $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t, s)x$ and

    a) $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t, s)x \equiv A(t)\frac{\partial}{\partial s}\mathcal{U}(t, s)x$ and the map $(t, s) \to A(t)\frac{\partial}{\partial s}\mathcal{U}(t, s)x$ is continuous.

    b) $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t, s)x \equiv \frac{\partial}{\partial t}\mathcal{U}(t, s)A(s)x$.

As an example for equation (2.1) one may consider the problem (e.g., [9])

$$\frac{\partial^2 z}{\partial t^2}(t, \tau) = \frac{\partial^2 z}{\partial \tau^2}(t, \tau) + a(t)\frac{\partial z}{\partial t}(t, \tau), \quad t \in [0, T], \tau \in [0, 2\pi],$$

$$z(t,0) = z(t,\pi) = 0, \quad \frac{\partial z}{\partial \tau}(t,0) = \frac{\partial z}{\partial \tau}(t,2\pi), \ t \in [0,T],$$

where $a(.) : I \to \mathbf{R}$ is a continuous function. This problem is modeled in the space $X = L^2(\mathbf{R},\mathbf{C})$ of 2π-periodic 2-integrable functions from $\mathbf{R}$ to $\mathbf{C}$, $A_1 z = \frac{d^2 z(\tau)}{d\tau^2}$ with domain $H^2(\mathbf{R},\mathbf{C})$ the Sobolev space of 2π-periodic functions whose derivatives belong to $L^2(\mathbf{R},\mathbf{C})$. It is well known that A_1 is the infinitesimal generator of strongly continuous cosine functions $C(t)$ on X. Moreover, A_1 has discrete spectrum; namely the spectrum of A_1 consists of eigenvalues $-n^2$, $n \in \mathbf{Z}$ with associated eigenvectors $z_n(\tau) = \frac{1}{\sqrt{2\pi}} e^{in\tau}$, $n \in \mathbf{N}$. The set z_n, $n \in \mathbf{N}$ is an orthonormal basis of X. In particular, $A_1 z = \sum_{n \in \mathbf{Z}} -n^2 < z, z_n > z_n$, $z \in D(A_1)$. The cosine function is given by $C(t)z = \sum_{n \in \mathbf{Z}} \cos(nt) < z, z_n > z_n$ with the associated sine function $S(t)z = t < z, z_0 > z_0 + \sum_{n \in \mathbf{Z}^*} \frac{\sin(nt)}{n} < z, z_n > z_n$.

For $t \in I$ define the operator $A_2(t)z = a(t)\frac{dz(\tau)}{d\tau}$ with domain $D(A_2(t)) = H^1(\mathbf{R},\mathbf{C})$. Set $A(t) = A_1 + A_2(t)$. It has been proved in [9] that this family generates an evolution operator as in Definition 2.1.

Definition 2.4. A continuous mapping $x(.) \in C(I,X)$ is called a mild solution of problem (1.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I,X)$ such that

$$f(t) \in F(t,x(t)) \quad a.e. \ (I), \tag{2.2}$$

$$x(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f(s)ds, \ t \in I. \tag{2.3}$$

We shall call $(x(.),f(.))$ a *trajectory-selection pair* of (1.1) if $f(.)$ verifies (2.2) and $x(.)$ is defined by (2.3).

Hypothesis 2.5. i) $F(.,.) : I \times X \to \mathcal{P}(X)$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(X)$ measurable.

ii) There exists $L(.) \in L^1(I,\mathbf{R}_+)$ such that, for any $t \in I, F(t,.)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t,x_1),F(t,x_2)) \leq L(t)\|x_1 - x_2\| \quad \forall x_1, x_2 \in X.$$

Hypothesis 2.6. Let S be a separable metric space, $X_0, X_1 \subset X$ are closed sets, $a_0(.) : S \to X_0$, $a_1(.) : S \to X_1$ and $c(.) : S \to (0,\infty)$ are given continuous mappings.

The continuous mappings $g(.) : S \to L^1(I,X)$, $y(.) : S \to C(I,X)$ are given such that

$$(y(s))''(t) = A(t)y(s)(t) + g(s)(t), \quad y(s)(0) \in X_0, \quad (y(s))'(0) \in X_1.$$

and there exists a continuous function $q(.) : S \to L^1(I, \mathbf{R}_+)$ such that

$$d(g(s)(t), F(t, y(s)(t))) \leq q(s)(t) \quad a.e.\,(I),\ \forall\, s \in S.$$

Theorem 2.7 ([7]). *Assume that Hypotheses 2.5 and 2.6 are satisfied. Then there exist $M > 0$ and the continuous functions $x(.) : S \to L^1(I, X)$, $h(.) : S \to C(I, X)$ such that for any $s \in S$ $(x(s)(.), h(s)(.))$ is a trajectory-selection pair of (1.1) satisfying for any $(t, s) \in I \times S$*

$$x(s)(0) = a_0(s), \quad (x(s))'(0) = a_1(s), \tag{2.4}$$

$$\begin{aligned} &\|x(s)(t) - y(s)(t)\| \leq \\ &M[c(s) + \|a_0(s) - y(s)(0)\| + \|a_1(s) - (y(s))'(0)\| + \int_0^t q(s)(u)du]. \end{aligned} \tag{2.5}$$

3   Main results

Our object of study is the reachable set of (1.1) defined by

$$R_F(T, X_0, X_1) := \{x(T); \quad x(.) \text{ is a mild solution of } (1.1)\}.$$

We consider a certain variational second-order differential inclusion and we shall prove that the reachable set of this variational inclusion from derived cones $C_0 \subset X$ to X_0 and $C_1 \subset X$ to X_1 at time T is a derived cone to the reachable set $R_F(T, X_0, X_1)$.

Throughout in this section we assume

Hypothesis 3.1. i) Hypothesis 2.5 is satisfied and $X_0, X_1 \subset X$ are closed sets.

ii) $(z(.), f(.)) \in C(I, X) \times L^1(I, X)$ is a trajectory-selection pair of (1.1) and a family $P(t, .) : X \to \mathcal{P}(X)$, $t \in I$ of convex processes satisfying the condition

$$P(t, u) \subset Q_{f(t)} F(t, .)(z(t); u) \quad \forall\, u \in dom(P(t, .)), \text{ a.e. } t \in I \tag{3.1}$$

is assumed to be given and defines the variational inclusion

$$v'' \in A(t)v + P(t, v). \tag{3.2}$$

Remark 3.2. We note that for any set-valued map $F(., .)$, one may find an infinite number of families of convex process $P(t, .)$, $t \in I$, satisfying condition (3.1); in fact any family of closed convex subcones of the quasitangent

cones, $\overline{P}(t) \subset Q_{(z(t),f(t))}\text{Graph}(F(t,.))$, defines the family of closed convex process

$$P(t,u) = \{v \in X; \ (u,v) \in \overline{P}(t)\}, \quad u,v \in X, \ t \in I$$

that satisfy condition (3.1). One is tempted, of course, to take as an "intrinsic" family of such closed convex process, for example Clarke's convex-valued directional derivatives $C_{f(t)}F(t,.)(z(t);.)$.

We recall (e.g. [1]) that, since $F(t,.)$ is assumed to be Lipschitz a.e. on I, the quasitangent directional derivative is given by

$$Q_{f(t)}F(t,.)((z(t);u)) = \{w \in X; \ \lim_{\theta \to 0+} \frac{1}{\theta}\mathrm{d}(f(t) + \theta w, F(t, z(t) + \theta u)) = 0\}. \tag{3.3}$$

We are able now to prove the main result of this paper.

Theorem 3.3. *Assume that Hypothesis 3.1 is satisfied, let $C_0 \subset X$ be a derived cone to X_0 at $z(0)$ and $C_1 \subset X$ be a derived cone to X_1 at $z'(0)$. Then the reachable set $R_P(T, C_0, C_1)$ of (3.2) is a derived cone to $R_F(T, X_0, X_1)$ at $z(T)$.*

Proof. In view of Definition 2.1, let $\{v_1, ..., v_m\} \subset R_P(T, C_0, C_1)$, hence such that there exist the trajectory-selection pairs $(u_1(.), g_1(.)), ..., (u_m(.), g_m(.))$ of the variational inclusion (3.2) such that

$$u_j(T) = v_j, \quad u_j(0) \in C_0, \quad u_j'(0) \in C_1, \quad j = 1, 2, ..., m \tag{3.4}$$

Since $C_0 \subset X$ is a derived cone to X_0 at $z(0)$ and $C_1 \subset X$ is a derived cone to X_1 at $z'(0)$, there exist the continuous mappings $a_0 : S = [0, \theta_0]^m \to X_0$, $a_1 : S \to X_1$ such that

$$\begin{aligned} a_0(0) &= z(0), \quad Da_0(0)s = \sum_{j=1}^m s_j u_j(0) \quad \forall s \in \mathbf{R}_+^m, \\ a_1(0) &= z'(0), \quad Da_1(0)s = \sum_{j=1}^m s_j u_j'(0) \quad \forall s \in \mathbf{R}_+^m. \end{aligned} \tag{3.5}$$

Further on, for any $s = (s_1, ..., s_m) \in S$ and $t \in I$ we denote

$$\begin{aligned} y(s)(t) &= z(t) + \sum_{j=1}^m s_j u_j(t), \\ g(s)(t) &= f(t) + \sum_{j=1}^m s_j g_j(t), \\ p(s)(t) &= \mathrm{d}(g(s)(t), F(t, y(s)(t))) \end{aligned} \tag{3.6}$$

and we prove that $y(.)$, $p(.)$ satisfy the hypothesis of Theorem 2.7.

Using the lipschitzianity of $F(t,.,.)$ we have that for any $s \in S$, the measurable function $p(s)(.)$ in (3.6) it is also integrable.

$$p(s)(t) = \mathrm{d}(g(s)(t), F(t, y(s)(t))) \leq \sum_{j=1}^m s_j \|g_j(t)\| + \mathrm{d}_H(F(t, z(t)), F(t, y(s)(t))) \leq \sum_{j=1}^m s_j \|g_j(t)\| + L(t)\sum_{j=1}^m s_j \|u_j(t)\|.$$

Moreover, the mapping $s \to p(s)(.) \in L^1(I, \mathbf{R}_+)$ is continuous (in fact Lipschitzian) since for any $s, s' \in S$ one may write succesively

$||p(s)(.) - p(s')(.)||_1 = \int_0^T ||p(s)(t) - p(s')(t)||dt \leq \int_0^T [||g(s)(t) - g(s')(t)||dt$
$+d_H(F(t, y(s)(t)), F(t, y(s')(t))))]dt \leq ||s - s'||(\sum_{j=1}^m \int_0^T [||g_j(t)|| +$
$+L(t)||u_j(t)||]dt)$

Let us define $S_1 := S \setminus \{(0, \ldots, 0)\}$ and $c(.) : S_1 \to (0, \infty)$, $c(s) := ||s||^2$. It follows from Theorem 2.7 the existence of a continuous function $x(.) : S_1 \to C(I, X)$ such that for any $s \in S_1$, $x(s)(.)$ is a mild solution of (1.1) with the properties (2.4)-(2.5).

For $s = 0$ we define $x(0)(t) = y(0)(t) = z(t)$ $\forall t \in I$. Obviously, $x(.) : S \to C(I, X)$ is also continuous.

Finally, we define the function $a(.) : S \to R_F(T, X_0, X_1)$ by

$$a(s) = x(s)(T) \quad \forall s \in S.$$

Obviously, $a(.)$ is continuous on S and satisfies $a(0) = z(T)$.

To end the proof we need to show that $a(.)$ is differentiable at $s_0 = 0 \in S$ and its derivative is given by

$$Da(0)(s) = \sum_{j=1}^m s_j v_j \quad \forall s \in \mathbf{R}_+^m$$

which is equivalent with the fact that

$$\lim_{s \to 0} \frac{1}{||s||}(||a(s) - a(0) - \sum_{j=1}^m s_j v_j||) = 0. \qquad (3.7)$$

From (2.7) we obtain

$$\frac{1}{||s||}||a(s) - a(0) - \sum_{j=1}^m s_j v_j|| \leq \frac{1}{||s||}||x(s)(T) - y(s)(T)|| \leq M||s|| + \frac{M}{||s||}||a_0(s) -$$

$$z(0) - \sum_{j=1}^m s_j u_j(0)|| + \frac{M}{||s||}||a_1(s) - z'(0) - \sum_{j=1}^m s_j u_j'(0)|| + M \int_0^T \frac{p(s)(u)}{||s||}du$$

and therefore in view of (3.5), relation (3.7) is implied by the following property of the mapping $p(.)$ in (3.6)

$$\lim_{s \to 0} \frac{p(s)(t)}{||s||} = 0 \quad \text{a.e. } (I). \qquad (3.8)$$

In order to prove the last property we note since $P(t, .)$ is a convex process for any $s \in S$ one has

$$\sum_{j=1}^{m} \frac{s_j}{||s||} g_j(t) \in P(t, \sum_{j=1}^{m} \frac{s_j}{||s||} u_j(t)) \subset Q_{f(t)} F(t, .)(z(t); \sum_{j=1}^{m} \frac{s_j}{||s||} u_j(t)) \quad \text{a.e. } (I).$$

Hence by (3.3) we obtain

$$\lim_{h \to 0+} \frac{1}{h} \mathrm{d}(f(t) + h \sum_{j=1}^{m} \frac{s_j}{||s||} g_j(t), F(t, z(t) + h \sum_{j=1}^{m} \frac{s_j}{||s||} u_j(t))) = 0. \quad (3.9)$$

In order to prove that (3.9) implies (3.8) we consider the compact metric space $S_+^{m-1} = \{\sigma \in \mathbf{R}_+^m; ||\sigma|| = 1\}$ and the real function $\phi_t(., .) : (0, \theta_0] \times S_+^{m-1} \to \mathbf{R}_+$ defined by

$$\phi_t(h, \sigma) = \frac{1}{h} \mathrm{d}(f(t) + h \sum_{j=1}^{m} \sigma_j g_j(t), F(t, z(t) + h \sum_{j=1}^{m} \sigma_j u_j(t))), \quad (3.10)$$

where $\sigma = (\sigma_1, ..., \sigma_m)$ and which according to (3.9) has the property

$$\lim_{\theta \to 0+} \phi_t(\theta, \sigma) = 0 \quad \forall \sigma \in S_+^{m-1} \text{ a.e. } (I) \quad (3.11)$$

Using the fact that $\phi_t(\theta, .)$ is Lipschitzian and the fact that S_+^{m-1} is a compact metric space, from (3.11) it follows that

$$\lim_{\theta \to 0+} \max_{\sigma \in S_+^{m-1}} \phi_t(\theta, \sigma) = 0$$

which implies the fact that

$$\lim_{s \to 0} \phi_t(||s||, \frac{s}{||s||}) = 0 \quad \text{a.e. } (I)$$

and the proof is complete.

An application of Theorem 3.3 concerns local controllability of the second-order differential inclusion in (1.1) along a reference trajectory, $z(.)$ at time T, in the sense that

$$z(T) \in \mathrm{int}(R_F(T, X_0, X_1)).$$

Theorem 3.4. *Let $X = \mathbf{R}^n$, $z(.), F(., .)$ and $P(., .)$ satisfy Hypothesis 3.1, let $C_0 \subset X$ be a derived cone to X_0 at $z(0)$ and $C_1 \subset X$ be a derived*

cone to X_1 at $z'(0)$. If the variational second-order differential inclusion in (3.2) is controllable at T in the sense that $R_P(T, C_0, C_1) = \mathbf{R}^n$, then the differential inclusion (1.1) is locally controllable along $z(.)$ at time T.

Proof. The proof is a straightforward application of Lemma 2.2 and of Theorem 3.3.

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