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OPTIMIZATION OF TEMPERATURE DISTRIBUTION IN TISSUE BY MICROWAVE INDUCED HEATING POWER

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Abstract. *An analytical study of optimal control problem in distributed parameter system, described by one- dimensional single layered (homogeneous) bio-heat equation, is investigated so as to attain a beneficial therapeutic desired rise of temperature at a particular point of location of the tumour inside the tissue at the end of operation of the process by controlling optimally time dependent heating power induced by microwave and also by controlling surface cooling temperature. The heating power is constructed according to the well-known Beer's Law [Karaa, Zhang and Yang, 2005]. The analytical investigation is carried out using "Maximal principle" [Pontrayagin et al., 1962] with a suitably constructed 'Hamiltonian function' by finite difference method. A numerical calculation of temperature distribution along the length of the tissue on various values of total time of operation of the process is computed.*

Keywords: optimal control, heat source, surface cooling temperature, tumour, hyperthermia

Notations

C	=	specific heat of tissue, J/(kg ^{o}C)
H	=	heat transfer coefficient between the skin and the ambient air, W m^{-2}/°C
k	=	thermal conductivity of tissue, W m^{-1}/°C
L	=	length of the tissue, m
x_1	=	position of tumour, m
χ	=	temperature, °C
χ_a	=	arterial temperature, °C
χ_0	=	initial temperature, °C
χ^*	=	desired temperature to be attained, °C
T	=	total time of the process, s

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$Q_1(x, t)$	=	spatial heating power induced by microwave, W m^{-3}
$u(t)$	=	surface cooling temperature, °C
$Q_2(t)$	=	time dependent heating Power, Q m^{-2}
B	=	scattering coefficient, m^{-1}
t_1	=	switching time of operation of $Q_2(t)$, s
t_2	=	switching time of operation of $u(t)$, s
ρ	=	density of tissue, kg m^{-3}
δ	=	Dirac – delta function.
ω	=	product of flow and heat capacity of blood, W m^{-3}/°C
Q_m	=	rate of metabolic heat generation, W m^{-3}

1. Introduction

In hyperthermia treatment, the temperature at the location of the tumour inside the tissue, described by bio-heat transfer equation, is raised to its therapeutic beneficial value while avoiding the damage of the healthy tissue due to overheating. In practice, microwave power level and cooling fluid temperature are the only variables which are accessible to direct control [Wagter, 1985]. It had been shown that surface cooling temperature could focus the microwave heating in deeper level of the tissue [Deng and Liu, 2002].

The fundamentals in the theory of optimal control of systems with distributed parameters were treated by [Butkovasky, 1969]. [Deng and Liu, 2002] obtained several analytical solutions to the bio-heat transfer problems with spatial heating on the skin surface using Green's function method. Here one of the solutions was applied to study the temperature distribution in one-dimensional homogeneous tissue due to the application of spatial heating power constructed from well-known Beer's law. [Dhar and Sinha, 1989] presented an analytical study on the strategy for achieving the desired rise of temperature of the tumour inside the tissue consisting of skin, fat, muscle and tumour layers by controlling surface cooling temperature with the finite difference method.

A distributed optimal control problem, so as to attain a desired beneficial temperature of the tumour inside the tissue due to induced microwave, was analytically studied by [Dhar and Dhar, 2010] under conjugate gradient method. [Dhar and Dhar, 2011] carried out a theoretical study on optimal control problem in hyperthermia by controlling microwave induced heating power in multi-layered tissue, where the 'Maximal Principle' was followed with a suitably constructed 'Hamiltonian' by the use of finite difference method. The study of [Dhar, Dhar and Dhar, 2011] on the optimal distribution of time dependent point heating power, induced by conducting heating probe at the tumour site, was of considerable importance in hyperthermia treatment.

[Karaa, Zhang and yang, 2005] presented a theoretical study on the temperature distribution in the tissue due to heating power induced by microwave, where the heat absorption in the muscle tissue was approximated by Beer's law. [Kuznestsov, 2006] studied analytically an optimal control problem in hyperthermia to attain the maximum rise of temperature on a located point of tumour at the end of operation of the process due to spatial heating.

[Loulou and Scott, 2002] studied analytically a control problem on optimization of thermal dose in hyperthermia by using the conjugate gradient method. A fundamental study on optimization problems were thoroughly discussed on the concept of 'Maximal Principle' in [Pontrayagin et al., 1962].

[Wagter, 1985] presented a computer simulation model for optimization of microwave-induced hyperthermia. Later, a simulation-oriented optimization method to determine the input control variables in hyperthermia was obtained by [Wagter, 1986].

In case of spatial heating power, one of the important issues is to deal with the most typical one where the heat flux decays exponentially with the distance from the surface of the tissue [Deng and Liu, 2002]. Such spatial heating power induced by microwave was, in fact, constructed by well- known Beer's law from which the heat distribution in the tissue could be well approximated [Karaa, Zhang and Yang, 2005].

In this paper, a distributed optimal control problem for a system described by Pennes bio-heat equation for a one-dimensional homogeneous tissue is analytically investigated such that a beneficial therapeutic desired temperature at a particular point of location of tumour $x = x_1$ inside the tissue can be achieved at the end of total time of operation of the process by controlling optimally time dependant heating power induced by microwave $Q_2(t)$ [Wm^{-2}] and also by controlling the surface cooling temperature $u(t)$ [°C]. Here, the spatial heating power $Q_1(x,t)$ induced by microwave has been constructed according to well-known Beer's law, given by, $Q_1(x,t)=\beta e^{-\beta x} Q_2(t)$ where $Q_2(t)$ [W m^{-2}] signifies time dependent heating power and β is scattering coefficient [Deng and Liu, 2002; Karaa, Zhang and Yang, 2005].

In course of analytical investigation of this problem, it is seen that the optimal control variables $Q_2(t)$ and $u(t)$ are singular controls. Thus, we have taken only two switching times t_1 and t_2, for the sake of simplicity, in the consideration of distributions of $Q_2(t)$ and $u(t)$ respectively. Here t_1 is the switching time during which the microwave induced heating power $Q_2(t)\,[\mathrm{Wm}^{-2}]$ operates in the intervals $(0, t_1)$ and (t_1, T); t_2 represents the switching time during which the surface cooling temperature $u(t)$ is operative in the intervals $(0, t_2)$ and (t_2, T). Here T signifies the total time of operation of the process.

The objective of this paper is to obtain the optimal values of controls $Q_2(t)$ and $u(t)$ for different specified values of t_1 and t_2 respectively [Wagter, 1985] using "Maximal principle" [Dhar and Sinha, 1989] with a suitably constructed 'Hamiltonian function' by finite difference method.

With these obtained values of $Q_2(t)$ and $u(t)$, the temperature distribution along the length of the tissue at different times against various values of total time of operation of the process T are obtained numerically which display the therapeutic beneficial desired temperature on the point of location of the tumour inside the tissue.

2. Mathematical analysis

The one dimensional Pennes bio-heat equation can be written as [Dhar and Dhar, 2011]:

$$\rho c \frac{\partial \chi}{\partial t} = k \frac{\partial^2 \chi}{\partial x^2} + \omega(\chi_a - \chi) + Q_1(x,t) + Qm \tag{1}$$

Where according to Beer's Law, $Q_1(x,t) = \beta e^{-\beta x} Q_2(t)$; $Q_2(t)$ is the time dependent microwave induced heating power and β signifies scattering coefficient.

Boundary condition:

$$k \frac{\partial \chi}{\partial x} = h\{\chi - u(t)\} \quad \text{on} \quad x = 0 \tag{2}$$

$$\chi = \chi_a \quad \text{on } x = L \tag{3}$$

initial condition:

$$\chi(x,o) = \chi_0 \tag{4}$$

Let us reduce the system with distributed parameters, given by equations (1)-(3), to lumped parameters described by ordinary differential equations of independent time variable, where finite difference method is used for dividing the segment $(0, L)$ on the X-axis into p equal number of equal sub intervals of length h for $h = \frac{L}{p}$, and the temperature of the tissue at each point $x = x_i$ is designated by quantities $q_i(t)$ $(i = 0, 1, 2, \dots p)$ [Butkovosky, 1969; Dhar and Dhar, 1989].

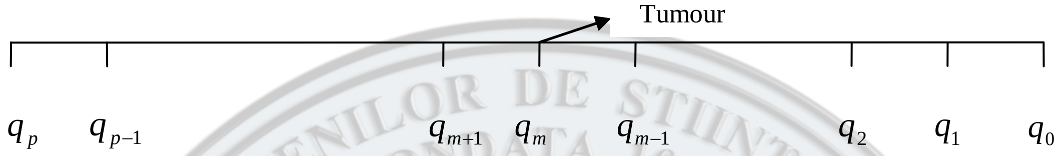

Fig. A.

Here, the temperature of the tumour at the point $x = x_1$ is represented by $q_m(t)$, given in figure A.

After discretising by finite difference method [Dhar and Sinha, 1989], the equation (1) stands as,

$$\frac{dq_1}{dt} = {}^1F_1$$

$$\frac{dq_j}{dt} = {}^2F_j, \; j = 2, 3, \dots, p-1 \tag{5}$$

The equations (2), (3), and (4) after discretising can be written as,

$$k\frac{q_1 - q_0}{h} = \lambda\{q_0 - u(t)\} \tag{6}$$

$$q_p = \chi_1 \tag{7}$$

and
$$q_j(0) = \chi_0, \; j = 1, 2, 3, \dots, p \tag{8}$$

where ${}^1F_1$ and ${}^2F_j$ in equation (5) thus stand as,

$${}^1F_1 = \frac{1}{\rho c}\left\{\frac{k^2}{h^2(k+\lambda h)} - \frac{2k}{h^2} - b\right\}q_1 + \frac{k}{\rho c h^2}q_2 + \frac{k_1\lambda u(t)}{\rho c h(k+\lambda h)} + \frac{Q_m}{\rho c} + \frac{\beta e^{-\beta h}Q_2(t)}{\rho c} + \frac{b}{\rho c}\chi_1 \tag{9}$$

$${}^2F_j = \frac{k}{\rho c h^2}\left(q_{j+1} - 2q_j + q_{j-1}\right) - \frac{b}{\rho c}\left(q_j - \chi_1\right) + \frac{Q_m}{\rho c} + \frac{\beta e^{-j\beta h}Q_2(t)}{\rho c} \tag{10}$$

which are obtained with the help of equations (5)-(7).

In order to attain a desired rise of temperature χ^* at a particular point of location of the tumour $x = x_1$ inside the tissue at the end of operation of the process *T* by controlling optimally microwave induced time dependent heating power $Q_2(t)$, [Wm^{-2}] and also by controlling surface cooling temperature $u(t)$ [°C], we would like to formulate the optimal control problem given as follows: to obtain the optimal control variables $Q_2(t)$, [Wm^{-2}] and $u(t)$ [°C] such that the quadratic deviation of the true temperature $\chi(x_1, T)$ from the desired temperature χ^* is minimized.

Thus the objective function after discretising [Butkovosky, 1969; Dhar and Sinha, 1989] stands,

$$\frac{1}{2}\left\{\chi^* - q_m(T)\right\}^2 \text{ which is to be minimized.} \tag{11}$$

Here, the temperature of the tumour on the point $x = x_1$ is represented by $q_m(t)$ (given in figure 1) and $q_m(T)$ designates the temperature on the point $x = x_1$ at the end of operation of the process *T*.

Following [Butkovosky, 1969; Dhar and Sinha, 1989], the Hamiltonian can be constructed as

$$H = \varphi_1{}^1F_1 + \sum_{j=2}^{p-1} \varphi_j{}^2F_j \tag{12}$$

where $\varphi_j(t)$ $(j = 1, 2, ..., p-1)$ are adjoint functions.

Let us write a functional *J*, given by,

$$J = -\frac{1}{2}\left\{\chi^* - q_m(T)\right\}^2 + \int_0^T \left\{H - \varphi_1 \frac{dq_1}{dt} - \sum_{j=2}^{p-1} \varphi_j \frac{dq_j}{dt}\right\} dt \tag{13}$$

we will represent first variation of functional *J* as δJ.

Thus, to obtain the optimality condition of control variables $Q_2(t)$ and $u(t)$, we consider the stationary condition $\delta J = 0$ for $\delta q_j(t)$, $\delta q_j(T)$ $(j = 1, 2, ..., p-1)$, $\delta Q_2(t)$ and $\delta u(t)$ by using calculus of variation and integrating by parts.

This leads to a system of adjoint functions $\varphi_j(t)$, given by

$$\frac{d}{dt}\varphi_j(t) = -\frac{\partial H}{\partial q_j}, \; (j = 1, 2, ..., p-1) \tag{14}$$

With terminal condition

$$\varphi_j(T) = 0\,,\ (j = 1, 2, ..., m-1, m+1, ..., p-1) \tag{15}$$

$$\varphi_m(T) = \chi^* - q_m(T) \tag{16}$$

The conditions of optimality of control variables $Q_2(t)$ [Wm^{-2}] and $u(t)$ [°C] stand as

$$\frac{\partial H}{\partial Q_2(t)} = 0 \tag{17}$$

and

$$\frac{\partial H}{\partial u(t)} = 0 \tag{18}$$

Thus,

$$Q_2(t) = \frac{\beta}{\rho c}\,\mathrm{sign}\sum_{j=1}^{p-1} e^{-j\beta h}\phi_j(t) \tag{19}$$

and

$$u(t) = \frac{k\lambda}{\rho ch(\lambda h + k)}\,\mathrm{sign}\,\phi_1(t) \tag{20}$$

From the equations (19) and (20) we note that $Q_2(t)$ and $u(t)$ are singular controls. Thus $Q_2(t)$ assumes extreme values within the intervals $(0, t_1)$ and (t_1, T) for switching time t_1. Similarly, the extreme values of $u(t)$ is operative within the intervals $(0, t_2)$ and (t_2, T) for switching time t_2.

Here it is assumed that the time control $Q_2(t)$ [Wm^{-2}] and $u(t)$ [°C] are piecewise constant functions of time that change values at certain specified discrete instants considered as switching times [Wagter, 1985].

As t_1 and t_2 are switching times of control variables $Q_2(t)$ and $u(t)$ respectively, the values of the optimal controls of $Q_2(t)$ and $u(t)$ can be obtained from the equations (19) and (20).

Thus,

$$\sum_{j=1}^{p-1} e^{-j\beta h}\varphi_j(t_1) = 0 \tag{21}$$

and

$$\varphi_1(t_2) = 0 \tag{22}$$

It is to note that since we are dealing with the problem of optimal heating of the tissue, the control variable $Q_2(t)$ must assume the maximum value on the first time segment $(0, t_1)$ and consequently maintain minimum value on the second time segment (t_1, T) [Butkovosky, 1962].

This aspect will make computer simulation one step forward towards finding the optimal values of control parameters $Q_2(t)$.

The value of optimal controls can be obtained with the help of equation (21) - (22) together with equations (8)-(22) by computer simulation [Dhar and Dhar, 2010].

3. Results and discussions

Data used in computation are given as follows [Dhar and Dhar, 2010]:

C = 3770 J kg^{-1} °C^{-1}

ρ = 998 kg m^{-3}

K = 0.5 W m^{-1} °C^{-1}

H = 6 W m^{-2} °C^{-1}

χ_a = 37 °C

χ^* = 43 °C

L = 0.01 m,

x_1 = 0.006 m

ω = 3000 W m^{-3} °C^{-1}

β = 200 m^{-1}

Q_m = 33800 W m^{-3}

χ_0 = 25 °C

T = 600 s, 800 s, 1000 s

The computation has been carried out for various values of total time of operation of the process T and accordingly optimal microwave heating power $Q_2(t)$ and $u(t)$ as well as the switching times t_1 and t_2 during which they are operative have been changed [Dhar and Dhar, 2010].

In the figures we have mentioned, $Q_2(t)$ [W m^{-2}] as $Q(t)$ [W m^{-2}].

From figure 1, it is observed that temperature of the tissue [°C] attains desired temperature 43 °C at the point of tumour x = 0.006 m at the end of the process T = 600 s for

$$Q(t) = 1764 \text{ Wm}^{-2}, 0 \le t \le 500 \text{ s}; Q(t) = 680 \text{ Wm}^{-2}, 500 \text{ s} \le t \le 600 \text{ s}$$

and $$u(t) = 9.25\,°\text{C}, 0 \le t \le 400 \text{ s}; u(t) = 6.25\,°C, 400 \text{ s} \le t \le 600 \text{s}.$$

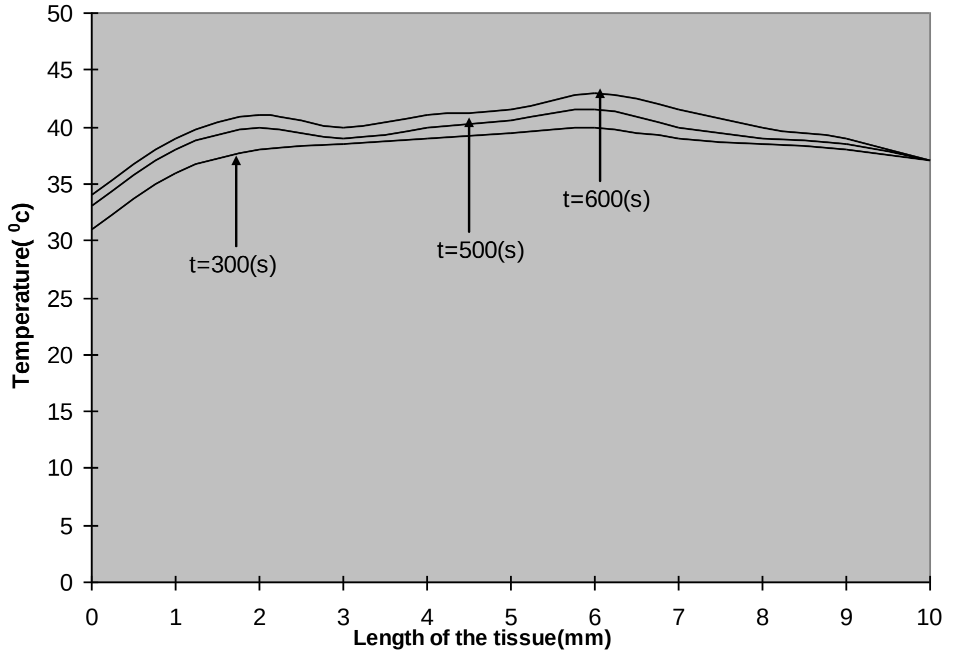

Fig 1 : Temperature(^{0}c) distribution of the tissue along the length of tissue(mm) for Q(t)=1764 wm^{-2}, 0<=t<=500; Q(t)=680 wm^{-2}, 500<= t<=600 and u(t)=9.25^{0}c, 0<=t<=400; u(t)=6.25^{0}c, 400<= t<=600

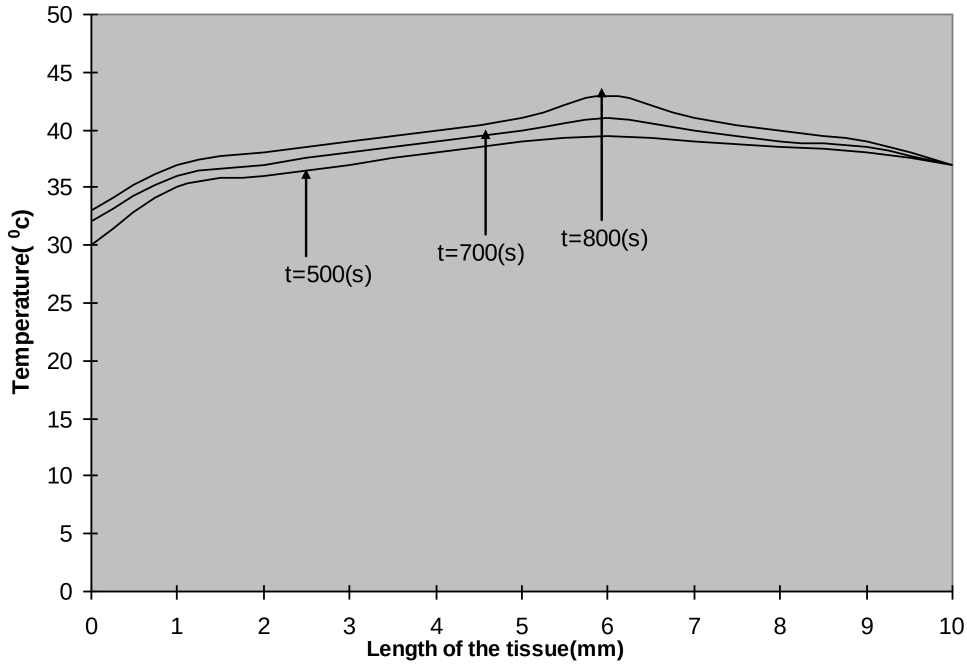

Fig 2 : Temperature(^{0}c) distribution of the tissue along the length of tissue(mm) for Q(t)=1525 wm^{-2}, 0<=t<=600; Q(t)=300 wm^{-2}, 600<= t<=800 and u(t)=10.59^{0}c, 0<=t<=500; u(t)=7.25^{0}c, 500<=t<=800

In figure 2, the temperature of the tissue (°C) at the point of tumour x = 0.006 m rises to desired temperature 43 °C at the end of duration of the process T = 800 s for the application of

$$Q(t) = 1525\,\mathrm{Wm}^{-2} = 0 \le t \le 600\ \mathrm{s};$$

$$Q(t) = 300\ \mathrm{Wm}^{-2},\ 600\mathrm{s} \le t \le 800\mathrm{s};$$

and
$$u(t) = 10.59\ °\mathrm{C},\ 0 \le t \le 500\mathrm{s};$$

$$u(t) = 7.25\ °\mathrm{C},\ 500\mathrm{s} \le t \le 800\mathrm{s}\ .$$

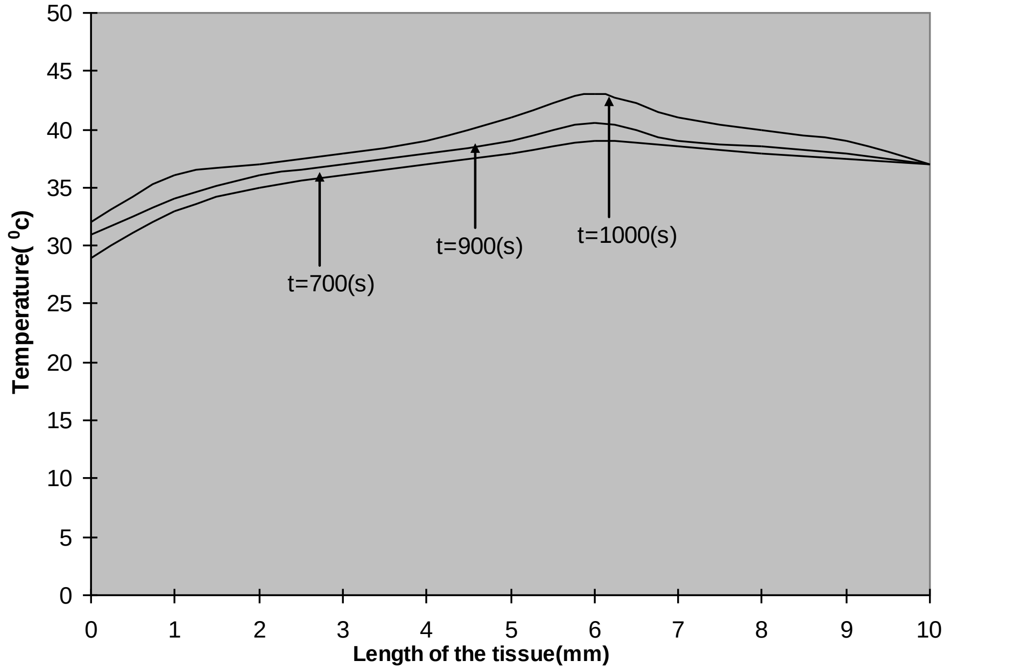

Fig 3 : Temperature(^{0}c) distribution of the tissue along the length of tissue(mm) for Q(t)=1280 wm^{-2}, 0<=t<=700; Q(t)=234 wm^{-2}, 700<= t<=1000 and u(t)=11.25^{0}c, 0<=t<=600; u(t)=8.59^{0}c, 600<=t<=1000

Figure 3 displays the temperature of the tissue [°C] along its length [mm] due to application of

$$Q(t) = 1280\ \mathrm{Wm}^{-2}\ ,\ 0 \le t \le 700\mathrm{s};\ \ Q(t) = 234\ \mathrm{Wm}^{-2}\ ,\ 700\mathrm{s} \le t \le 1000\mathrm{s}$$

and

$$u(t) = 11.25\ °\mathrm{C},\ 0 \le t \le 600\mathrm{s};\ \ u(t) = 8.59\ °\mathrm{C},\ 600\mathrm{s} \le t \le 1000\mathrm{s}\ .$$

Here, the desired temperature 43 °C is attained on the tumour location x = 0.006 m at the end of operation of the forces T = 1000 s.

4. Conclusions

It is observed that the temperature of tissue increases on the left side of the tumour at $x = 6$ mm till it attains the beneficial desired temperature $43\,°\mathrm{C}$ at the end of operation of the process and then the temperature of the tissue on the right side of the tumour decreases steadily to $37\,°\mathrm{C}$ (arterial temperature). Further, it is to note that as the total time of operation of the process increases from $T = 600$ s to 1000 s, the first time segment of operation of the process $(0, t_1)$ increases with the corresponding decrease of $Q(t)$ $[\mathrm{Wm}^{-2}]$ in this segment for the switching time t_1. The surface cooling temperature $u(t)$ [°C] increases in the first time segment of operation of the process $(0, t_2)$ as the total time of operation increases from $T = 600$ s to 1000 s for the switching time t_2.

Again, it is seen that the temperature of the healthy tissue on the both sides of the tumour are less than desired rise of temperature $43\,°\mathrm{C}$ and thus the damage of the healthy tissue is avoided due to overheating.

This analytical study of the optimal control problem may be used in case of computer- aided therapy planning in hyperthermia treatment. It can further be developed at different points of location of the tumour along with different length of tissue which may focus as a useful guideline to illustrate the versatility of the computer program.

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