AN "AVERAGE" CONCEPT IN CALCULATING THE ELASTIC COEFFICIENTS FOR COMPOSITE MATERIALS

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Rezumat. *Această lucrare propune o metodă nouă, originală constând în considerarea unui concept de „medie" în calcularea componentelor matricei rigidităţii în cazul barei compozite, cu două faze, când proprietăţile elastice ale constituenţilor sunt cunoscute.*

Abstract. *This work proposes a new and original method consisting in considering an "average" concept in calculating the components of the rigidity matrix in case of two phases right composite bars, when the elastic properties of their constituents are known.*

Keywords: Elastic coefficients, Composite materials, Elastic properties, Industrial domains

1. Introduction

The composite materials are very useful in many industrial domains like aircraft and automotive. The elastic coefficients are determined mostly using experimental ways. A real challenge is the analytical calculus of these coefficients.

Preliminaries

Let's consider a composite material made of some many distinct constituents (phases). Those phases could present different forms of anisotropy.
Many authors [1], [2] take into account that under the action of certain external charges the material will accumulate for each phase:
This way, the S_{ik} surface between the "i" phase and "k" phase (fig. 1) the continuity conditions are:
- for the strain-stress status:

$$\sigma_{nn}^{(i)} = \sigma_{nn}^{(k)};\ \sigma_{nt}^{(i)} = \sigma_{nt}^{(k)};\ \sigma_{n\tau}^{(i)} = \sigma_{n\tau}^{(k)}; \qquad (1)$$

- for the deformation status:

$$\varepsilon_{tt}^{(i)} = \varepsilon_{tt}^{(k)};\ \varepsilon_{\tau\tau}^{(i)} = \varepsilon_{\tau\tau}^{(k)};\ \varepsilon_{t\tau}^{(i)} = \varepsilon_{t\tau}^{(k)} \text{ so that } \gamma_{t\tau}^{(i)} = \gamma_{t\tau}^{(k)}; \qquad (2)$$

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where $\sigma_{nn};\ \sigma_{nt};\ \sigma_{n\tau}$ are the efforts and $\varepsilon_{tt};\ \varepsilon_{\tau\tau};\ \varepsilon_{t\tau}=\frac{1}{2}\gamma_{t\tau}$ are the deformations corresponding to three ortogonal directions $\vec{n}$; $\vec{t}$; $\vec{\tau}$ the way that:

$\vec{n}$ the normal unit vector to S_{ik}, from the "k" phase to the "i" phase.

$\vec{t}$ and $\vec{\tau}$ are unit vectors normal each other and they are situated in the tangent plane to S_{ik} ($\vec{t}\times\vec{\tau}=\vec{n}$).

The composite material can be assumed as being a continuum material.

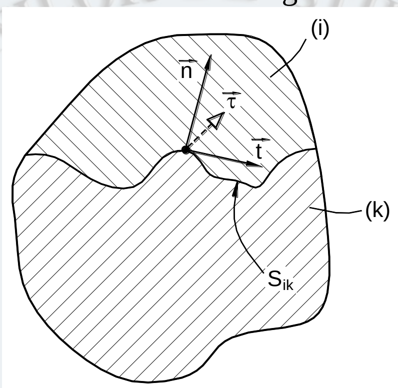

Fig. 1

An "average" concept for the calculus of the elastic coefficients for a two-phases composite material

The basic concept of this workpaper [3], [4] is that if a composite has constituents having not too much different characteristics then a certain external charge will produce an unique deformation status as well as an unique strain-stress status that are parts of an unique constitutive equation. Both deformation status and strain-stress status have components with values that are to be calculated like some sort of averages of the values provided by $\{\sigma^{(i)}\}$ and $\{\varepsilon^{(i)}\}$ that characterizes each and every "*i*" phase. This way, the material an be considered as being "homogenous" and it remains to determine its type of anisotropy.

We'll define:

- the vector of average deformations $\{\bar{\varepsilon}\}$:

$$\{\bar{\varepsilon}\}=\frac{1}{V}\sum_{k=1}^{n}\iiint_{V_k}\{\varepsilon^{(k)}\}dV\,; \tag{3}$$

- the vector of average internal efforts $\{\bar{\sigma}\}$:

$$\{\bar{\sigma}\}=\frac{1}{V}\sum_{k=1}^{n}\iiint_{V_k}\{\sigma^{(k)}\}dV\,; \tag{4}$$

Where V is the volume of the composite, dV is the elementar volume $dV = dx_1 dx_2 dx_3$, V_k being the volume of the "k" phase $\left(V = \sum_{k=1}^{n} V_k \right)$.

Considering all these, the constitutive equation of the composite as a whole can be written as follows:

$$\{\overline{\sigma}\} = [E]\{\overline{\varepsilon}\}; \tag{5}$$

or:

$$\{\overline{\varepsilon}\} = [C]\{\overline{\sigma}\}; \tag{6}$$

where [E] and [C] are, respectively the rigidity matrix and the thickness matrix of the composite material. We’ll show that both matrices are symmetric, having their main diagonal different from zero, so they are inversable.

In order to be more more specific, let’s consider a composite material presenting two homogenous and isotropic phases. In its non-deformed status the material is considered as having its own reference system $O x_1 x_2 x_3$. We note S_{12} the separation surface between these two surfaces. We are to consider also a curvilinear coordinates system chosen the way that the in each and every point of S_{12} the unit vectors corresponding to this system are actually the same with the normal unit vector an other two rectangular unit vectors placed in the tangent plane to S_{12}. We’ll note these $\vec{n}$; $\vec{t}$ and $\vec{\tau}$. If $\vec{i_1}$ is the unit vector of $O x_1$; $\vec{i_2}$ is the unit vector of $O x_2$, and $\vec{i_3}$ is unit vector of $O x_3$, so we have:

$$\vec{n} = \alpha_1 \vec{i_1} + \beta_1 \vec{i_2} + \delta_1 \vec{i_3}\ ;\ \vec{t} = \alpha_2 \vec{i_1} + \beta_2 \vec{i_2} + \delta_2 \vec{i_3}\ ;\ \vec{\tau} = \alpha_3 \vec{i_1} + \beta_3 \vec{i_2} + \delta_3 \vec{i_3}\ . \tag{7}$$

Obviously, from (18) it results that the matrix of changing the unit vector basis from $Ox_1x_2x_3$ to the reference system attached to curvilinear coordinates will be:

$$[S] = \begin{bmatrix} \alpha_1 & \beta_1 & \delta_1 \\ \alpha_2 & \beta_2 & \delta_2 \\ \alpha_3 & \beta_3 & \delta_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \delta_1 & \delta_2 & \delta_3 \end{bmatrix}^t ; \tag{8}$$

Let’s consider:

$$[\sigma_n] = \begin{bmatrix} \sigma_{nn} & \sigma_{nt} & \sigma_{n\tau} \\ \sigma_{nt} & \sigma_{tt} & \sigma_{t\tau} \\ \sigma_{n\tau} & \sigma_{t\tau} & \sigma_{\tau\tau} \end{bmatrix} ; \quad [\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} ; \tag{9}$$

the (symmetric) matrices attached to stress-strain tensors in the local reference system defined by those curvilinear coordinates and in the $O x_1 x_2 x_3$ reference system.

Let’s also consider:

$$[\varepsilon_n]=\begin{bmatrix}\varepsilon_{nn} & \frac{1}{2}\gamma_{nt} & \frac{1}{2}\gamma_{n\tau}\\ \frac{1}{2}\gamma_{nt} & \varepsilon_{tt} & \frac{1}{2}\gamma_{t\tau}\\ \frac{1}{2}\gamma_{n\tau} & \frac{1}{2}\gamma_{t\tau} & \varepsilon_{\tau\tau}\end{bmatrix};\quad [\varepsilon]=\begin{bmatrix}\varepsilon_{11} & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13}\\ \frac{1}{2}\gamma_{12} & \varepsilon_{22} & \frac{1}{2}\gamma_{23}\\ \frac{1}{2}\gamma_{13} & \frac{1}{2}\gamma_{23} & \varepsilon_{33}\end{bmatrix}; \tag{10}$$

the (symmetric) matrices attached to deformation tensors in those two reference system. Taking into account the tensorial character of the matrices from (9) and (10), the following relations does exist:

$$[\sigma]=[S]^t[\sigma_n][S];\ [\sigma_n]=[S][\sigma][S]^t; \tag{11}$$

and:

$$[\varepsilon]=[S]^t[\varepsilon_n][S];\ [\varepsilon_n]=[S][\varepsilon][S]^t; \tag{12}$$

and, also:

$$[S]^{-1}=[S]^t; \tag{13}$$

[S] being an eigen matrix. Given all that we can write:

$$\{\sigma\}'=[\alpha]\{\sigma_n\}';\ \{\sigma_n\}'=[\alpha]^{-1}\{\sigma\}'; \tag{14}$$

$$\{\varepsilon\}'=[\beta]\{\varepsilon_n\}';\ \{\varepsilon_n\}'=[\beta]^{-1}\{\varepsilon\}'; \tag{15}$$

where:

$$\{\sigma\}'=\{\sigma_{11};\sigma_{22};\sigma_{33};\sigma_{12};\sigma_{13};\sigma_{23}\}^t;\ \{\sigma_n\}'=\{\sigma_{nn};\sigma_{tt};\sigma_{\tau\tau};\sigma_{tn};\sigma_{n\tau};\sigma_{\tau t}\}^t; \tag{16}$$

$$\{\varepsilon\}'=\{\varepsilon_{11};\varepsilon_{22};\varepsilon_{33};\gamma_{12};\gamma_{13};\gamma_{23}\}^t;\ \{\varepsilon_n\}'=\{\varepsilon_{nn};\varepsilon_{tt};\varepsilon_{\tau\tau};\varepsilon_{tn};\varepsilon_{n\tau};\varepsilon_{\tau t}\}^t; \tag{17}$$

Concerning $\{\sigma\}$ and $\{\varepsilon\}$, notations (16) and (17) are marking the order difference between the components 4 and 6 versus the usual notations and well known in literature. Concerning $\{\sigma_n\}$ and $\{\varepsilon_n\}$ we notice the same difference with respect to notation suggested by $[\sigma_n]$ given by (9) and $[\varepsilon_n]$ given (10) which is to be:

$$\{\sigma_n\}=\{\sigma_{nn};\sigma_{tt};\sigma_{\tau\tau};\sigma_{\tau t};\sigma_{n\tau};\sigma_{tn}\}^t;\ \{\varepsilon_n\}=\{\varepsilon_{nn};\varepsilon_{tt};\varepsilon_{\tau\tau};\varepsilon_{\tau t};\varepsilon_{n\tau};\varepsilon_{tn}\}^t; \tag{18}$$

In order to find out the matrix of elastic coefficients we assume that, on the action of a certain external charge an internal status of strain-stress and deformation will appear in both phases. The column vectors attached to each status can be arranged in the curvilinear coordinates reference system as it follows:

$$\{\varepsilon_n^{(1)}\}=\begin{Bmatrix}\{\varepsilon_n^*\}\\ \{\varepsilon_{n_1}\}\end{Bmatrix};\ \{\varepsilon_n^{(2)}\}=\begin{Bmatrix}\{\varepsilon_n^*\}\\ \{\varepsilon_{n_2}\}\end{Bmatrix}; \tag{19}$$

$$\{\sigma_n^{(1)}\}=\begin{Bmatrix}\{\sigma_{n_1}\}\\ \{\sigma_n^*\}\end{Bmatrix};\ \{\sigma_n^{(2)}\}=\begin{Bmatrix}\{\sigma_{n_2}\}\\ \{\sigma_n^*\}\end{Bmatrix}; \tag{20}$$

where:

$$\left\{\varepsilon_n^*\right\}=\begin{Bmatrix}\varepsilon_{tt}^{(1)}\\ \varepsilon_{\tau\tau}^{(1)}\\ \gamma_{t\tau}^{(1)}\end{Bmatrix}=\begin{Bmatrix}\varepsilon_{tt}^{(2)}\\ \varepsilon_{\tau\tau}^{(2)}\\ \gamma_{t\tau}^{(2)}\end{Bmatrix}=\begin{Bmatrix}\varepsilon_{tt}\\ \varepsilon_{\tau\tau}\\ \gamma_{t\tau}\end{Bmatrix}; \tag{21}$$

and:

$$\left\{\varepsilon_{n_1}\right\}=\left\{\gamma_{nt}^{(1)};\gamma_{n\tau}^{(1)};\varepsilon_{nn}^{(1)}\right\}^t;\quad \left\{\varepsilon_{n_2}\right\}=\left\{\gamma_{nt}^{(2)};\gamma_{n\tau}^{(2)};\varepsilon_{nn}^{(2)}\right\}^t; \tag{22}$$

and:

$$\left\{\sigma_n^*\right\}=\begin{Bmatrix}\sigma_{nt}^{(1)}\\ \sigma_{n\tau}^{(1)}\\ \sigma_{nn}^{(1)}\end{Bmatrix}=\begin{Bmatrix}\sigma_{nt}^{(2)}\\ \sigma_{n\tau}^{(2)}\\ \sigma_{nn}^{(2)}\end{Bmatrix}=\begin{Bmatrix}\sigma_{nt}\\ \sigma_{n\tau}\\ \sigma_{nn}\end{Bmatrix}; \tag{23}$$

and, also:

$$\left\{\sigma_{n_1}\right\}=\left\{\sigma_{tt}^{(1)};\sigma_{\tau\tau}^{(1)};\sigma_{t\tau}^{(1)}\right\}^t;\quad \left\{\sigma_{n_2}\right\}=\left\{\sigma_{tt}^{(2)};\sigma_{\tau\tau}^{(2)};\sigma_{t\tau}^{(2)}\right\}^t. \tag{24}$$

Considering all that we’ll come back to the two phases-composite and we can write for each phase:

$$\begin{Bmatrix}\left\{\varepsilon_n^*\right\}\\ \left\{\varepsilon_{n_1}\right\}\end{Bmatrix}=\begin{bmatrix}[A_1] & [B_1]\\ [B_1]^t & [C_1]\end{bmatrix}\begin{Bmatrix}\left\{\sigma_{n_1}\right\}\\ \left\{\sigma_n^*\right\}\end{Bmatrix}; \tag{25}$$

and:

$$\begin{Bmatrix}\left\{\varepsilon_n^*\right\}\\ \left\{\varepsilon_{n_2}\right\}\end{Bmatrix}=\begin{bmatrix}[A_2] & [B_2]\\ [B_2]^t & [C_2]\end{bmatrix}\begin{Bmatrix}\left\{\sigma_{n_2}\right\}\\ \left\{\sigma_n^*\right\}\end{Bmatrix}; \tag{26}$$

where, materials being homogenous, we have noted the following matrices:

$$[A_i]=\begin{bmatrix}\frac{1}{E_i} & -\frac{\upsilon_i}{E_i} & 0\\ -\frac{\upsilon_i}{E_i} & \frac{1}{E_i} & 0\\ 0 & 0 & \frac{1}{G_i}\end{bmatrix};\ [A_i]=\begin{bmatrix}0 & 0 & -\frac{\upsilon_i}{E_i}\\ 0 & 0 & -\frac{\upsilon_i}{E_i}\\ 0 & 0 & 0\end{bmatrix};\ [C_i]=\begin{bmatrix}\frac{1}{G_i} & 0 & 0\\ 0 & \frac{1}{G_i} & 0\\ 0 & 0 & \frac{1}{G_i}\end{bmatrix};\mathrm{i}=\overline{1,2} \tag{27}$$

This way the constitutive equations for both phases take the following form: - for the first constituent:

$$\left\{\varepsilon_n^*\right\}=[A_1]\left\{\sigma_{n_1}\right\}+[B_1]\left\{\sigma_n^*\right\};\quad \left\{\varepsilon_{n_1}\right\}=[B_1]^t\left\{\sigma_{n_1}\right\}+[C_1]\left\{\sigma_n^*\right\}; \tag{28}$$

- for the second constituent:

$$\left\{\varepsilon_n^*\right\}=[A_2]\left\{\sigma_{n_2}\right\}+[B_2]\left\{\sigma_n^*\right\};\quad \left\{\varepsilon_{n_2}\right\}=[B_2]^t\left\{\sigma_{n_2}\right\}+[C_2]\left\{\sigma_n^*\right\}. \tag{29}$$

From (28) and (29) we conclude:

$$\left\{\sigma_{n_1}\right\}=[A_1]^{-1}\left\{\varepsilon_n^*\right\}-[A_1]^{-1}[B_1]\left\{\sigma_n^*\right\};$$

$$\left\{\varepsilon_{n_1}\right\}=\left[B_1\right]^t\left[A_1\right]^{-1}\left\{\varepsilon_n^*\right\}+\left[\left[C_1\right]-\left[B_1\right]^t\left[A_1\right]^{-1}\left[B_1\right]\right]\left\{\sigma_n^*\right\}; \tag{30}$$

and:

$$\left\{\sigma_{n_2}\right\}=\left[A_2\right]^{-1}\left\{\varepsilon_n^*\right\}-\left[A_2\right]^{-1}\left[B_2\right]\left\{\sigma_n^*\right\};$$

$$\left\{\varepsilon_{n_2}\right\}=\left[B_2\right]^t\left[A_{21}\right]^{-1}\left\{\varepsilon_n^*\right\}+\left[\left[C_2\right]-\left[B_2\right]^t\left[A_2\right]^{-1}\left[B_2\right]\right]\left\{\sigma_n^*\right\}. \tag{31}$$

Putting (30) and (31) into (29) and (20) we'll have:

$$\left\{\varepsilon_n^{(1)}\right\}=\begin{bmatrix}[I] & [0]\\ [D_1] & [L_1]\end{bmatrix}\begin{Bmatrix}\left\{\varepsilon_n^*\right\}\\ \left\{\sigma_n^*\right\}\end{Bmatrix} \tag{32}$$

$$\left\{\sigma_n^{(1)}\right\}=\begin{bmatrix}[T_1] & [U_1]\\ [0] & [I]\end{bmatrix}\begin{Bmatrix}\left\{\varepsilon_n^*\right\}\\ \left\{\sigma_n^*\right\}\end{Bmatrix}; \tag{33}$$

and:

$$\left\{\varepsilon_n^{(2)}\right\}=\begin{bmatrix}[I] & [0]\\ [D_2] & [L_2]\end{bmatrix}\begin{Bmatrix}\left\{\varepsilon_n^*\right\}\\ \left\{\sigma_n^*\right\}\end{Bmatrix}; \tag{34}$$

$$\left\{\sigma_n^{(2)}\right\}=\begin{bmatrix}[T_2] & [U_2]\\ [0] & [I]\end{bmatrix}\begin{Bmatrix}\left\{\varepsilon_n^*\right\}\\ \left\{\sigma_n^*\right\}\end{Bmatrix}; \tag{35}$$

where we used the following notations:

$$\left[D_1\right]=\left[B_1\right]^t\left[A_1\right]^{-1}\ ;\ \sum_{k=1}^{2}\left[\left[P_k\right]+\left[R_k\right]\cdot\left[D_k\right]\right]=\left[B'\right]\cdot\left[\sum_{k=1}^{2}\left[E_k\right]\cdot\left[T_k\right]\right]+\left[C\right]\cdot\left[\sum_{k=1}^{2}\left[G_k\right]\cdot\left[T_k\right]\right]; \tag{36}$$

$$[I]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix};\ [0]=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}; \tag{37}$$

$$\left[D_2\right]=\left[B_2\right]^t\left[A_2\right]^{-1};\ \left[L_2\right]=\left[C_2\right]-\left[B_2\right]^t\left[A_2\right]^{-1}\left[B_2\right]; \tag{38}$$

$$\left[T_1\right]=\left[A_1\right]^{-1};\ \left[U_1\right]=-\left[A_1\right]^{-1}\left[B_1\right]; \tag{39}$$

$$\left[T_2\right]=\left[A_2\right]^{-1};\ \left[U_2\right]=-\left[A_2\right]^{-1}\left[B_2\right]; \tag{40}$$

Using (14) and (16) we can determine the column vectors for the strain-stress status and for the deformation status in the $O x_1 x_2 x_3$ reference system:

$$\left\{\varepsilon^{(1)}\right\}=\begin{bmatrix}[M] & [N]\\ [P] & [R]\end{bmatrix}\begin{bmatrix}[I] & [O]\\ [D_1] & [L_1]\end{bmatrix}\begin{Bmatrix}\left\{\varepsilon_n^*\right\}\\ \left\{\sigma_n^*\right\}\end{Bmatrix};\ \left\{\varepsilon^{(2)}\right\}=\begin{bmatrix}[M] & [N]\\ [P] & [R]\end{bmatrix}\begin{bmatrix}[I] & [O]\\ [D_2] & [L_2]\end{bmatrix}\begin{Bmatrix}\left\{\varepsilon_n^*\right\}\\ \left\{\sigma_n^*\right\}\end{Bmatrix}; \tag{41}$$

and:

$$\left\{\sigma^{(1)}\right\}=\begin{bmatrix}[E] & [F]\\ [G] & [H]\end{bmatrix}\begin{bmatrix}[T_1] & [U_1]\\ [0] & [I]\end{bmatrix}\begin{Bmatrix}\left\{\varepsilon_n^*\right\}\\ \left\{\sigma_n^*\right\}\end{Bmatrix};$$

$$\left\{\sigma^{(2)}\right\}=\begin{bmatrix}[E] & [F]\\ [G] & [H]\end{bmatrix}\begin{bmatrix}[T_2] & [U_2]\\ [0] & [I]\end{bmatrix}\begin{Bmatrix}\left\{\varepsilon_n^*\right\}\\ \left\{\sigma_n^*\right\}\end{Bmatrix}; \tag{42}$$

Where, taking into account the concrete form for [α] and [β]:

$$[M]=\begin{bmatrix} \alpha_2^2 & \alpha_3^2 & \alpha_2\alpha_3 \\ \beta_2^2 & \beta_3^2 & \beta_2\beta_3 \\ 2\alpha_2\beta_2 & 2\alpha_3\beta_3 & \alpha_2\beta_3+\alpha_3\beta_2 \end{bmatrix};$$

$$[N]=\begin{bmatrix} \alpha_1\alpha_2 & \alpha_1\alpha_3 & \alpha_1^2 \\ \beta_1\beta_2 & \beta_1\beta_3 & \beta_1^2 \\ \alpha_1\beta_2+\alpha_2\beta_1 & \alpha_1\beta_3+\alpha_3\beta_1 & 2\alpha_1\beta_1 \end{bmatrix}; \tag{43}$$

$$[P]=\begin{bmatrix} 2\alpha_2\delta_2 & 2\alpha_3\delta_3 & \alpha_2\delta_3+\alpha_3\delta_2 \\ 2\beta_2\delta_2 & 2\beta_3\delta_3 & \beta_2\delta_3+\beta_3\delta_2 \\ \delta_2^2 & \delta_3^2 & \delta_2\delta_3 \end{bmatrix};$$

$$[R]=\begin{bmatrix} \alpha_1\delta_2+\alpha_2\delta_1 & \alpha_1\delta_3+\alpha_3\delta_1 & 2\alpha_1\delta_1 \\ \beta_1\delta_2+\beta_2\delta_1 & \beta_1\delta_3+\beta_3\delta_1 & 2\beta_1\delta_1 \\ \delta_1\delta_2 & \delta_1\delta_3 & \delta_1^2 \end{bmatrix}. \tag{44}$$

and:

$$[E]=\begin{bmatrix} \alpha_2^2 & \alpha_3^2 & 2\alpha_2\alpha_3 \\ \beta_2^2 & \beta_3^2 & 2\beta_2\beta_3 \\ \alpha_2\beta_2 & \alpha_3\beta_3 & \alpha_2\beta_3+\alpha_3\beta_2 \end{bmatrix};$$

$$[F]=\begin{bmatrix} 2\alpha_1\alpha_2 & 2\alpha_1\alpha_3 & \alpha_1^2 \\ 2\beta_1\beta_2 & 2\beta_1\beta_3 & \beta_1^2 \\ \alpha_1\beta_2+\alpha_2\beta_1 & \alpha_1\beta_3+\alpha_3\beta_1 & \alpha_1\beta_1 \end{bmatrix}; \tag{45}$$

$$[G]=\begin{bmatrix} \alpha_2\delta_2 & \alpha_3\delta_3 & \alpha_2\delta_3+\alpha_3\delta_2 \\ \beta_2\delta_2 & \beta_3\delta_3 & \beta_2\delta_3+\beta_3\delta_2 \\ \delta_2^2 & \delta_3^2 & 2\delta_2\delta_3 \end{bmatrix};$$

$$[H]=\begin{bmatrix} \alpha_1\delta_2+\alpha_2\delta_1 & \alpha_1\delta_3+\alpha_3\delta_1 & \alpha_1\delta_1 \\ \beta_1\delta_2+\beta_2\delta_1 & \beta_1\delta_3+\beta_3\delta_1 & \beta_1\delta_1 \\ 2\delta_1\delta_2 & 2\delta_1\delta_3 & \delta_1^2 \end{bmatrix}. \tag{46}$$

The column vector of medium deformations in the $Ox_1\,x_2\,x_3$ results from (3) and the column vector of the medium strain-stress status results from (4). Particularly, if $\{\varepsilon_n\}$ and $\{\sigma_n\}$ have constant components we obtain:

$$\left[\sum_{k=1}^{2}\begin{bmatrix} [M_k] & [N_k] \\ [P_k] & [R_k] \end{bmatrix}\begin{bmatrix} [I] & [0] \\ [D_k] & [L_k] \end{bmatrix}\right]\begin{Bmatrix} \{\varepsilon_n^*\} \\ \{\sigma_n^*\} \end{Bmatrix}; \tag{47}$$

$$\left[\sum_{k=1}^{2}\begin{bmatrix}[E_k] & [F_k]\\ [G_k] & [H_k]\end{bmatrix}\begin{bmatrix}[T_k] & [U_k]\\ [0] & [I]\end{bmatrix}\right]\begin{Bmatrix}\{\varepsilon_n^*\}\\ \{\sigma_n^*\}\end{Bmatrix}; \tag{48}$$

where:

$$[M_k]=\frac{1}{V}\iiint_{V_k}[M]dV\,;\ [E_k]=\frac{1}{V}\iiint_{V_k}[E]dV\,;\ [N_k]=\frac{1}{V}\iiint_{V_k}[N]dV\,;$$

$$[F_k]=\frac{1}{V}\iiint_{V_k}[F]dV\,;$$

$$[P_k]=\frac{1}{V}\iiint_{V_k}[P]dV\,;[G_k]=\frac{1}{V}\iiint_{V_k}[G]dV\,;[R_k]=\frac{1}{V}\iiint_{V_k}[R]dV\,;[H_k]=\frac{1}{V}\iiint_{V_k}[H]dV \tag{49}$$

The constitutive equation for the composite as a whole in the $Ox_1x_2x_3$ reference system will take the following form:

$$\{\bar{\varepsilon}\}=\begin{bmatrix}[A] & [B]\\ [B'] & [C]\end{bmatrix}\{\bar{\sigma}\}; \tag{50}$$

Putting in (50) $\{\bar{\varepsilon}\}$ given by (47) and $\{\bar{\sigma}\}$ given by (48) we'll obtain:

$$\sum_{k=1}^{2}[[M_k]+[N_k]\cdot[D_k]]=[A]\cdot\left[\sum_{k=1}^{2}[E_k]\cdot[T_k]\right]+[B]\cdot\left[\sum_{k=1}^{2}[G_k]\cdot[T_k]\right]; \tag{51}$$

$$\sum_{k=1}^{2}[N_k]\cdot[L_k]=[A]\cdot\left[\sum_{k=1}^{2}[[F_k]+[E_k]\cdot[U_k]]\right]+[B]\cdot\left[\sum_{k=1}^{2}[[H_k]+[G_k]\cdot[U_k]]\right]; \tag{52}$$

$$\sum_{k=1}^{2}[[P_k]+[R_k]\cdot[D_k]]=[B']\cdot\left[\sum_{k=1}^{2}[E_k]\cdot[T_k]\right]+[C]\cdot\left[\sum_{k=1}^{2}[G_k]\cdot[T_k]\right]; \tag{53}$$

$$\sum_{k=1}^{2}[R_k]\cdot[L_k]=[B']\cdot\left[\sum_{k=1}^{2}[[F_k]+[E_k]\cdot[U_k]]\right]+[C]\cdot\left[\sum_{k=1}^{2}[[H_k]+[G_k]\cdot[U_k]]\right]; \tag{54}$$

The rigidity matrix contains both matrices [A] and [B] given by (51) and (52) and contains also both matrices [B'] and [C] given by (53) and (54).

We obtain:

$$[A]=\left[\left[\sum_{k=1}^{2}[[M_k]+[N_k]\cdot[D_k]]\right]\cdot\left[\sum_{k=1}^{2}[G_k]\cdot[T_k]\right]^{-1}-\left[\sum_{k=1}^{2}[N_k]\cdot[L_k]\right]\cdot\right.$$
$$\left.\cdot\left[\sum_{k=1}^{2}[[H_k]+[G_k]\cdot[H_k]]\right]^{-1}\right]\cdot\left[\left[\sum_{k=1}^{2}[E_k]\cdot[T_k]\right]\cdot\left[\sum_{k=1}^{2}[G_k]\cdot[T_k]\right]^{-1}-\right.$$
$$\left.-\left[\sum_{k=1}^{2}[[F_k]+[E_k]\cdot[U_k]]\right]\cdot\left[\sum_{k=1}^{2}[[H_k]+[G_k]\cdot[H_k]]\right]^{-1}\right]^{-1} \tag{55}$$

$$[B]=\left[\left[\sum_{k=1}^{2}[[M_k]+[N_k]\cdot[D_k]]\right]\cdot\left[\sum_{k=1}^{2}[E_k]\cdot[T_k]\right]^{-1}-\left[\sum_{k=1}^{2}[N_k]\cdot[L_k]\right]\cdot\right.$$
$$\cdot\left.\left[\sum_{k=1}^{2}[[F_k]+[E_k]\cdot[U_k]]\right]^{-1}\right]\cdot\left[\left[\sum_{k=1}^{2}[G_k]\cdot[T_k]\right]\cdot\left[\sum_{k=1}^{2}[E_k]\cdot[T_k]\right]^{-1}-\right.$$
$$\left.-\left[\sum_{k=1}^{2}[[H_k]+[G_k]\cdot[U_k]]\right]\cdot\left[\sum_{k=1}^{2}[[F_k]+[E_k]\cdot[U_k]]\right]^{-1}\right]^{-1} \tag{56}$$

$$[B']=\left[\left[\sum_{k=1}^{2}[[P_k]+[R_k]\cdot[D_k]]\right]\cdot\left[\sum_{k=1}^{2}[G_k]\cdot[T_k]\right]^{-1}-\left[\sum_{k=1}^{2}[R_k]\cdot[L_k]\right]\cdot\right.$$
$$\cdot\left.\left[\sum_{k=1}^{2}[[H_k]+[G_k]\cdot[U_k]]\right]^{-1}\right]\cdot\left[\left[\sum_{k=1}^{2}[E_k]\cdot[T_k]\right]\cdot\left[\sum_{k=1}^{2}[G_k]\cdot[T_k]\right]^{-1}-\right.$$
$$\left.-\left[\sum_{k=1}^{2}[[F_k]+[E_k]\cdot[U_k]]\right]\cdot\left[\sum_{k=1}^{2}[[H_k]+[G_k]\cdot[U_k]]\right]^{-1}\right]^{-1} \tag{57}$$

$$[C]=\left[\left[\sum_{k=1}^{2}[[P_k]+[R_k]\cdot[D_k]]\right]\cdot\left[\sum_{k=1}^{2}[E_k]\cdot[T_k]\right]^{-1}-\left[\sum_{k=1}^{2}[R_k]\cdot[L_k]\right]\cdot\right.$$
$$\cdot\left.\left[\sum_{k=1}^{2}[[F_k]+[E_k]\cdot[U_k]]\right]^{-1}\right]\cdot\left[\left[\sum_{k=1}^{2}[G_k]\cdot[T_k]\right]\cdot\left[\sum_{k=1}^{2}[E_k]\cdot[T_k]\right]^{-1}-\right.$$
$$\left.-\left[\sum_{k=1}^{2}[[H_k]+[G_k]\cdot[U_k]]\right]\cdot\left[\sum_{k=1}^{2}[[F_k]+[E_k]\cdot[U_k]]\right]^{-1}\right]^{-1} \tag{58}$$

In case that these matrices are inversable the problem has an unique solution.

Let’s consider that [3], [4] constituents are disposed like fig. 2 shows, the constituents being 1 and 2.

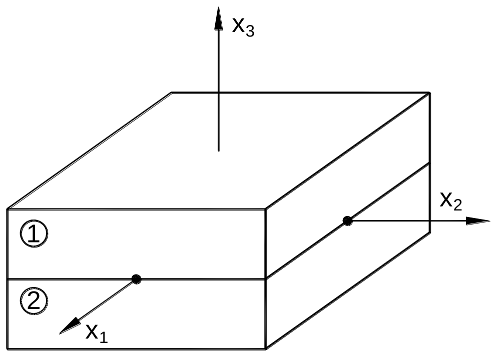

Fig. 2

The unit vector $\vec{n}$ to the separation surface will have the Ox_3 direction, $\vec{t}$ will have the Ox_1 direction and $\vec{\tau}$ will have the Ox_2 direction, so:

$$\vec{n}=\vec{i}_3;\ (\alpha_1=0;\beta_1=0;\delta_1=1);\ \vec{t}=\vec{i}_1;\ (\alpha_2=1;\beta_2=0;\delta_2=0);\ \vec{\tau}=\vec{i}_2;\ (\alpha_3=0;\beta_3=1;\delta_3=0). \quad (59)$$

$$[M]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix};[N]=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix};[P]=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix};[R]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}; \quad (60)$$

$$[E]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix};[F]=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix};[G]=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix};[H]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}; \quad (61)$$

With all that, we'll have:

$$\sum_{k=1}^{2}\left[[M_k]+[N_k]\cdot[D_k]\right]=[I];\ \sum_{k=1}^{2}[G_k]\cdot[T_k]=[0];$$

$$\sum_{k=1}^{2}[E_k]\cdot[T_k]=v_1[A_1]^{-1}+v_2[A_2]^{-1}\overset{not.}{=}[\varphi];\ \sum_{k=1}^{2}[N_k]\cdot[L_k]=[0],$$

$$\sum_{k=1}^{2}\left[[F_k]+[G_k]\cdot[U_k]\right]=-\left[v_1[A_1]^{-1}[B_1]+v_2[A_2]^{-1}[B_2]\right]\overset{not.}{=}-[\psi]; \quad (62)$$

$$\sum_{k=1}^{2}\left[[H_k]+[G_k]\cdot[U_k]\right]=[I];$$

$$\sum_{k=1}^{2}\left[[P_k]+[R_k]\cdot[D_k]\right]=v_1[B_1]^t[A_1]^{-1}+v_2[B_2]^t[A_2]^{-1}=[\psi]^t;$$

$$\sum_{k=1}^{2}[R_k]\cdot[L_k]=v_1\left[[C_1]-[B_1]^t[A_1]^{-1}[B_1]\right]+v_2\left[[C_2]-[B_2]^t[A_2]^{-1}[B_2]\right]\overset{not.}{=}[\xi].$$

This way (51); (52); (53); (54) takes the form:

$$[I]=[A][\varphi];\ [0]=-[A][\psi]+[B];\ [\psi]^t=[B'][\varphi];\ [\xi]=-[B'][\psi]+[C]. \quad (63)$$

So:

$$[A]=[\varphi]^{-1}=\left[v_1[A_1]^{-1}+v_2[A_2]^{-1}\right]^{-1}; \quad (64)$$

$$[B]=[\varphi]^{-1}[\psi]=\left[v_1[A_1]^{-1}+v_2[A_2]^{-1}\right]^{-1}\cdot\left[v_1[A_1]^{-1}[B_1]+v_2[A_2]^{-1}[B_2]\right]^{-1}; \quad \mathbf{(65)}$$

$$[B']=[\psi]^t[\varphi]^{-1}=\left[v_1[B_1]^t[A_1]^{-1}+v_2[B_2]^t[A_2]^{-1}\right]\cdot\left[v_1[A_1]^{-1}+v_2[A_2]^{-1}\right]^{-1}; \quad \mathbf{(66)}$$

$$[C]=[\xi]+[\psi]^t[\varphi]^{-1}[\psi]=v_1\left[[C_1]-[B_1]^t[A_1]^{-1}[B_1]\right]+$$
$$+v_2\left[[C_2]-[B_2]^t[A_2]^{-1}[B_2]\right]+\left[v_1[B_1]^t[A_1]^{-1}+v_2[B_2]^t[A_2]^{-1}\right]\cdot$$
$$\cdot\left[v_1[A_1]^{-1}+v_2[A_2]^{-1}\right]^{-1}\cdot\left[v_1[A_1]^{-1}[B_1]+v_2[A_2]^{-1}[B_2]\right]. \quad (67)$$

Where v_1 and v_2 are the volumic ratios for both phases. The matrices $[A_1]$ and $[A_2]$ are symmetric. It's easy to verify that:

$$[A]^t = [A];\ [B'] = [B]^t;\ [C]^t = [C]. \tag{68}$$

So, the rigidity matrix of the composite material is symmetric and that confirms once more the validity of the hypothesis adopted.

So, we have:

$$E_1^{(c)} = E_2^{(c)} = \frac{v_1^2 E_1^2 \left(1-\upsilon_2^2\right) + v_2^2 E_2^2 \left(1-\upsilon_2^2\right) + 2 v_1 v_2 E_1 E_2 \left(1-\upsilon_1 \upsilon_2\right)}{v_1 E_1 \left(1-\upsilon_2^2\right) + v_2 E_2 \left(1-\upsilon_1^2\right)}; \tag{69}$$

$$\upsilon_{12}^{(c)} = \upsilon_{21}^{(c)} = \frac{v_1 \upsilon_1 E_1 \left(1-\upsilon_2^2\right) + v_2 \upsilon_2 E_2 \left(1-\upsilon_1^2\right)}{v_1 E_1 \left(1-\upsilon_2^2\right) + v_2 E_2 \left(1-\upsilon_1^2\right)}; \tag{70}$$

$$G_{12}^{(c)} = v_1 G_1 + v_2 G_2; \tag{71}$$

$$\upsilon_{13}^{(c)} = \upsilon_{31}^{(c)} = \frac{\left[\upsilon_1 v_1 \left(1-\upsilon_2\right) + \upsilon_2 v_2 \left(1-\upsilon_1\right)\right] \cdot \left[v_1 E_1 \left(1+\upsilon_2\right) + v_2 E_2 \left(1+\upsilon_1\right)\right]}{v_1 E_1 \left(1-\upsilon_2^2\right) + v_2 E_2 \left(1-\upsilon_1^2\right)}; \tag{72}$$

$$G_{13}^{(c)} = G_{23}^{(c)} = \frac{G_1 G_2}{G_1 v_2 + G_2 v_1}; \tag{73}$$

$$\frac{1}{E_3^{(c)}} = \frac{2\left[\upsilon_1 v_1 \left(1-\upsilon_2\right) + \upsilon_2 v_2 \left(1-\upsilon_1\right)\right]^2 \cdot \left[v_1 E_1 \left(1+\upsilon_2\right) + v_2 E_2 \left(1+\upsilon_1\right)\right]}{\left(1-\upsilon_1\right)\left(1-\upsilon_2\right)\left[v_1^2 E_1^2 \left(1-\upsilon_2^2\right) + v_2^2 E_2^2 \left(1-\upsilon_1^2\right) + 2 v_1 v_2 E_1 E_2 \left(1-\upsilon_1 \upsilon_2\right)\right]} +$$

$$+ \frac{v_1 \left(1+\upsilon_1\right)\left(1-2\upsilon_1\right)}{\left(1-\upsilon_1\right) E_1} + \frac{v_2 \left(1+\upsilon_2\right)\left(1-2\upsilon_2\right)}{\left(1-\upsilon_2\right) E_2}; \tag{74}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \\ \gamma_{31} \\ \gamma_{32} \\ \varepsilon_{33} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1^{(c)}} & -\frac{\upsilon_{12}^{(c)}}{E_1^{(c)}} & 0 & 0 & 0 & -\frac{\upsilon_{13}^{(c)}}{E_1} \\ -\frac{\upsilon_{21}^{(c)}}{E_2^{(c)}} & \frac{1}{E_2^{(c)}} & 0 & 0 & 0 & -\frac{\upsilon_{23}^{(c)}}{E_2} \\ 0 & 0 & \frac{1}{G_{12}^{(c)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{13}^{(c)}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{23}^{(c)}} & 0 \\ -\frac{\upsilon_{13}^{(c)}}{E_1} & -\frac{\upsilon_{23}^{(c)}}{E_2} & 0 & 0 & 0 & \frac{1}{E_3^{(c)}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{Bmatrix}; \tag{75}$$

We have to specify that the "(c)" index refers to the entire composite.

Finaly, we have, writing (75) in its usual form:

$$\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{32}\\ \gamma_{31}\\ \gamma_{21}\end{Bmatrix}=\begin{bmatrix}\frac{1}{E_1^{(c)}} & -\frac{\upsilon_{12}^{(c)}}{E_1^{(c)}} & -\frac{\upsilon_{13}^{(c)}}{E_1} & 0 & 0 & 0\\ -\frac{\upsilon_{21}^{(c)}}{E_2^{(c)}} & \frac{1}{E_2^{(c)}} & -\frac{\upsilon_{23}^{(c)}}{E_2} & 0 & 0 & 0\\ -\frac{\upsilon_{13}^{(c)}}{E_1} & -\frac{\upsilon_{23}^{(c)}}{E_2} & \frac{1}{E_3^{(c)}} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G_{23}^{(c)}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}^{(c)}} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}^{(c)}}\end{bmatrix}\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{32}\\ \sigma_{31}\\ \sigma_{21}\end{Bmatrix}; \qquad (76)$$

Conclusions

The conclusion is that in this case the composite material can be considered as being homogenous and orthotropic having the planes Ox_1x_2; Ox_1x_3; Ox_2x_3 as planes of elastic symmetry.

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