SYNTHESIS AND POLYPHASE IMPLEMENTATION OF WIDE-BAND LOW-PASS CIRCULAR 2D FIR FILTERS

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Rezumat. *Se propune o tehnică de proiectare analitică în domeniul frecvenţă pentru filtre 2D de tip FIR trece jos de bandă largă cu simetrie circulară. Proiectarea este bazată pe un fitru prototip trece-jos, ideal de maxim plat, cu banda de trecere specificată, derivat din funcţia tangentă hiperbolică. Prototipul ideal este aproximat cu un polinom trigonometric, printr-o schimbare de variabilă şi dezvoltare în serie Chebyshev, astfel rezultând răspunsul în frecvenţă factorizat al prototipului 1D. Aplicand prototipului o transformare de frecvenţă specifică, derivată din transformarea generală McClellan, va rezulta direct răspunsul în frecvenţă al filtrului 2D circular dorit, de asemenea factorizat. Filtrul proiectat are o formă precisă, cu distorsiuni neglijabile. S-a propus de asemenea şi o implementare eficientă la nivel de sistem pentru filtrul proiectat, utilizând metoda de descompunere polifazică şi filtrare pe blocuri, ce duce la o structură de filtrare eficientă, cu complexitate aritmetică redusă şi un grad ridicat de paralelism.*

Abstract. An analytical synthesis technique in the frequency domain is proposed for a particular class of 2D filters, namely circular wide-band low-pass FIR filters. The design starts from a low-pass prototype filter which is ideally maximally flat, with specified bandwidth, based on hyperbolic tangent function. This is approximated as a trigonometric polynomial using a change of variable and the Chebyshev series, thus obtaining the factored frequency response of the FIR filter prototype. Applying a specific frequency mapping derived from the more general McClellan transform, the frequency response of the desired circular 2D FIR filter results directly, also in factored form. The designed filter has an accurate shape, with negligible distortions. We also proposed a computationally efficient implementation at system level, based on polyphase decomposition and block filtering approach, which leads to a filtering structure with low aritmetic complexity and a high degree of parallelism.

Keywords: 2D FIR filters, circular filters, approximations, analytic design, polyphase filtering

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Introduction

Synthesis techniques and implementation structures for two-dimensional filters are fundamental research topics in the vast domain of digital signal processing.

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Some currently used design methods rely on global numerical optimization, while others are analytical, employing some frequency transformations applied to various 1D filter prototypes, deriving directly the 2D filter with a desired frequency response [1]. A specific tool for 2D filter design is the McClellan transform [2-4]. A specific class of 2D filters, widely used in image processing, are circular filters, of either FIR and IIR type, developed in early papers like [5-7]. Some works such as [8, 9] elaborate novel techniques regarding both design and implementation of circular filters. As regards implementation aspects, there have been proposed various 2D filter structures, both for FIR and IIR versions. The fast block implementation of 2D digital FIR filters was proposed in early papers like [10, 11]. A high performance 2D parallel block-filtering system for real-time applications was presented in [12]. Previous related papers of the authors on the design and implementation of some circular filters are [13-15].

In this work we propose an analytical synthesis technique for 2D circular FIR filters. The method is based on a prototype filter, to which a specific 1D to 2D frequency transformation is subsequently applied. We have chosen a maximally-flat, smooth function, easy to approximate by a trigonometric polynomial, as shown next. Then, an efficient system-level implementation was developed for the designed filter, using a polyphase decomposition of a 2D filtering operation with a large kernel size.

Wide-band low-pass zero-phase prototype for 2D circular filters

As prototype for the desired 2D circular filter, we choose a continuous, smooth function with a high steepness, namely the hyperbolic tangent. Specifically, one ideal function for a low-pass maximally-flat filter may have the expression:

$$H_I(\omega) = 0.5 \cdot \left(\tanh\left(10 \cdot \left(\omega + \pi/2\right)\right) - \tanh\left(10 \cdot \left(\omega - \pi/2\right)\right)\right) \tag{1}$$

Now we look for a trigonometric series expansion of the ideal function $H_I(\omega)$, which has to be an approximation as accurate as possible on the frequency range $[-\pi, \pi]$. The most convenient for our purpose is the Chebyshev series expansion, because it yields an efficient approximation of a given function, which is uniform along the desired interval. The Chebyshev series of a given function on a specified interval can be easily found using a symbolic computation software like MAPLE. However, we will need rather a trigonometric expansion of $H_I(\omega)$, namely in $\cos(n\omega)$, rather than a polynomial in the frequency variable ω. Therefore, prior to Chebyshev series calculation, the following variable change will be applied:

$$\omega = \arccos\left(x/\pi\right) \Leftrightarrow x = \pi\cos(\omega) \tag{2}$$

and thus a general, ideal prototype $H_I(\omega)$ can be further written in intermediate variable x as $H_I(\omega) = H_I(\arccos(x/\pi))$, then the Chebyshev series approximation is found as the polynomial expansion in variable x:

$$H_I(x) \cong H_P(x) = \sum_{n=0}^{N} c_n \cdot x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_N x^N \tag{3}$$

This results indirectly, using the change of variable (2). For the specific prototype (1), the following intermediate function in variable x results:

$$H_I(x) = 0.5 \cdot \left(\tanh\left(10 \cdot \left(\arccos\left(x/\pi\right) + \pi/2\right)\right) - \tanh\left(10 \cdot \left(\arccos\left(x/\pi\right) - \pi/2\right)\right)\right) \tag{4}$$

As an odd-parity function, it has the following Chebyshev series approximation, in odd powers of x:

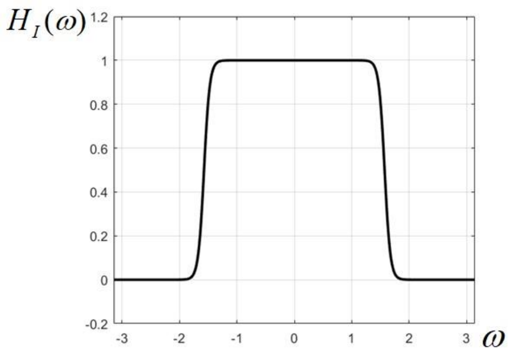
$$\begin{aligned} H_I(x) \cong\ & 0.499726 + 1.086074 \cdot x - 0.910301 \cdot x^3 + 0.441843 \cdot x^5 \\ & - 0.113842 \cdot x^7 + 0.015767 \cdot x^9 - 0.0011094 \cdot x^{11} + 0.00003111 \cdot x^{13} \end{aligned} \tag{5}$$

The number of terms N=13 is chosen to ensure a desired precision (specified by the maximum error, imposed of value 0.06 in our case).

Next, substituting back $x = \pi \cos(\omega)$, the following frequency response of the low-pass prototype filter, in factored form, is derived, where $y = \cos\omega$:

$$\begin{aligned} & H_P(\omega) = 89.741 \cdot (y + 0.99053)(y + 0.943854)(y + 0.849386)(y + 0.71926) \\ & (y + 0.553156)(y + 0.369463)(y + 0.2044424) \cdot (y^2 - 0.87852 \cdot y + 0.28381) \\ & (y^2 - 1.65878 \cdot y + 0.7454)(y^2 - 2.09279 \cdot y + 1.102997) \end{aligned} \tag{6}$$

This prototype has the frequency response shown in Figure 1, b and we notice it has a relatively small ripple in the passband and stopband.



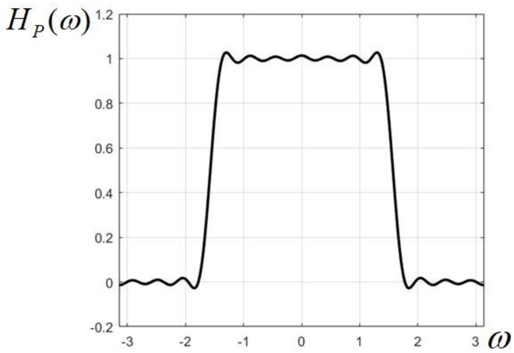


Fig. 1. Wide-band low-pass prototype filter

a. Ideal prototype based on hyperbolic tangent function

b. Approximated prototype filter

Design of circularly-symmetric 2D FIR filters

As shown before in a particular case, substituting back $x = \pi\cos(\omega)$ in the polynomial expression (3), we obtain an approximation $H_P(\omega)$ of the given ideal prototype $H_I(\omega)$, as a trigonometric polynomial in powers of $\cos\omega$ [15]:

$$H_I(\omega) \cong H_P(\omega) = \sum_{n=0}^{N} b_n \cdot \cos^n(\omega) = b_0 + b_1\cos\omega + b_2\cos^2\omega + b_3\cos^3\omega + ... + b_N\cos^N\omega \quad (7)$$

with $b_0 = c_0$ and $b_k = c_k \cdot \pi^k$. According to the fundamental theorem of algebra, the polynomial (7) can be factored into first and second order polynomials in $\cos\omega$, as follows (where $n + 2m = N$, the filter order):

$$H(\omega) = k \cdot \prod_{i=1}^{n}(\cos\omega + a_i) \cdot \prod_{j=1}^{m}(\cos^2\omega + a_{1j}\cos\omega + a_{2j}) \quad (8)$$

We describe in this section an efficient design technique for 2D circularly-symmetric filters, based on the previous 1D filters, considered as prototypes. Given a prototype with transfer function $H_P(\omega)$, the 2D circular filter function $H_C(\omega_1, \omega_2)$ results applying the mapping $\omega \to \sqrt{\omega_1^2 + \omega_2^2}$:

$$H_C(\omega_1, \omega_2) = H_P\left(\sqrt{\omega_1^2 + \omega_2^2}\right) \quad \mathbf{(9)}$$

The currently-used approximation of the 2D circular cosine function $\cos\sqrt{\omega_1^2 + \omega_2^2}$ is given by the 3×3 array [4]:

$$\mathbf{C} = \begin{bmatrix} 0.125 & 0.25 & 0.125 \\ 0.25 & -0.5 & 0.25 \\ 0.125 & 0.25 & 0.125 \end{bmatrix} \quad (10)$$

such that we have the following approximation, a particular case of the McClellan transform [2-4]:

$$\cos\sqrt{\omega_1^2 + \omega_2^2} \cong C(\omega_1, \omega_2) = -0.5 + 0.5\left(\cos\omega_1 + \cos\omega_2\right) + 0.5\cos\omega_1 \cdot \cos\omega_2 \quad (11)$$

In order to obtain a filter with circular symmetry from the factored 1D prototype function, we simply replace $\cos\omega$ with the circular cosine function (11) in the prototype frequency response (8). Corresponding to this factored expression, the filter convolution kernel **A** results as:

$$\mathbf{A} = k \cdot \mathbf{M}_1 * \mathbf{M}_2 * \ldots * \mathbf{M}_n * \mathbf{N}_1 * \mathbf{N}_2 * \ldots * \mathbf{N}_m \quad (12)$$

where $\mathbf{M}_i$ ($i = 1\ldots n$) are 3×3 arrays and $\mathbf{N}_j$ ($j = 1\ldots m$) are 5×5 arrays, given by the expressions:

$$\mathbf{M}_i = \mathbf{C} + a_i \cdot \mathbf{A}_{01} \tag{13}$$

$$\mathbf{N}_j = \mathbf{C} * \mathbf{C} + a_{1j} \cdot \mathbf{C}_0 + a_{2j} \cdot \mathbf{A}_{02} \tag{14}$$

where $\mathbf{A}_{01}$ is a 3×3 zero array and $\mathbf{A}_{02}$ is a 5×5 zero array with the central element equal to one; $\mathbf{C}_0$ is a 5×5 array obtained by bordering $\mathbf{C}$ (3×3) with zeros. The above expressions correspond to the factors in expression (8).
In our case, the component matrices (7 matrices of size 3x3 and 3 matrices of size 5x5) result according to expressions (13) and (14). For instance, $\mathbf{M}_1$ is given by:

$$\mathbf{M}_1 = \begin{bmatrix} 0.125 & 0.25 & 0.125 \\ 0.25 & 0.49053 & 0.25 \\ 0.125 & 0.25 & 0.125 \end{bmatrix} \tag{15}$$

The other 3x3 matrices, $\mathbf{M}_2$ - $\mathbf{M}_7$ derive from matrix **C** given by (10), substituting the center element with the values, respectively: 0.443855, 0.349386, 0.21926, 0.053156, -0.130537, -0.295557. The 5x5 matrices $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ are the following:

$$\mathbf{N}_1 = \begin{bmatrix} 0.015625 & 0.0625 & 0.09375 & 0.0625 & 0.015625 \\ 0.0625 & -0.10981 & -0.34463 & -0.10981 & 0.0625 \\ 0.09375 & -0.34463 & 1.285567 & -0.34463 & 0.09375 \\ 0.0625 & -0.10981 & -0.34463 & -0.10981 & 0.0625 \\ 0.015625 & 0.0625 & 0.09375 & 0.0625 & 0.015625 \end{bmatrix} \tag{16}$$

$$\mathbf{N}_2 = \begin{bmatrix} 0.015625 & 0.0625 & 0.09375 & 0.0625 & 0.015625 \\ 0.0625 & -0.20735 & -0.53969 & -0.20735 & 0.0625 \\ 0.09375 & -0.53969 & 2.13729 & -0.53969 & 0.09375 \\ 0.0625 & -0.20735 & -0.53969 & -0.20735 & 0.0625 \\ 0.015625 & 0.0625 & 0.09375 & 0.0625 & 0.015625 \end{bmatrix} \tag{17}$$

$$\mathbf{N}_3 = \begin{bmatrix} 0.015625 & 0.0625 & 0.09375 & 0.0625 & 0.015625 \\ 0.0625 & -0.26159 & -0.64819 & -0.26159 & 0.0625 \\ 0.09375 & -0.64819 & 2.71189 & -0.64819 & 0.09375 \\ 0.0625 & -0.26159 & -0.64819 & -0.26159 & 0.0625 \\ 0.015625 & 0.0625 & 0.09375 & 0.0625 & 0.015625 \end{bmatrix} \tag{18}$$

The frequency response and its corresponding contour plot for the designed circular wide-band low-pass 2D FIR filter, derived by frequency mapping from the prototype in Figure 1(b), with the specified bandwidth, are displayed in Figure 2. It can be noticed that the filter has a relatively precise shape in the frequency plane and a given ripple in both the pass band and the stop band, corresponding to the imposed specifications.

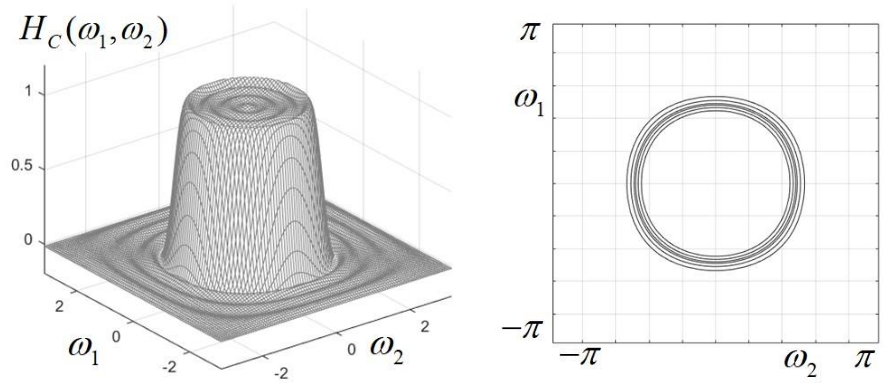


Fig. 2. Wide-band circular FIR filter

a. Frequency response of the 2D circular FIR filter

b. Corresponding contour plot in the frequency plane

Polyphase implementation of the 2D circular filter

In this section an efficient implementation of the designed FIR circular filter is proposed, based on a polyphase decomposition of a 2D filtering operation with a large convolution kernel (27x27). In order to tackle this high complexity problem we employ a block processing approach [16] and a polyphase decomposition. Using related ideas as in [13], we have first derived a 2D 3x3 filtering algorithm that is presented as follows.

To achieve this task, the kernel of the 2D filter obtained from design and the input image to be filtered are decimated by factors 3 and 9, respectively; after this, a polyphase filtering technique is used. Through this approach, three partial output component images are calculated in parallel, namely $\mathbf{Y}_0$, $\mathbf{Y}_1$ and $\mathbf{Y}_2$ given by equations (19), (21) and (23):

$$\mathbf{Y}_0 = \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_0 \\ \mathbf{O}_{3,6} & \mathbf{A}_0 \\ \mathbf{O}_{3,6} & \mathbf{A}_0 \end{bmatrix} \cdot diag(V_0) \times diag\left(\begin{bmatrix} \mathbf{A}_0^T & \mathbf{O}_{6,3} & \mathbf{O}_{6,3} \\ \mathbf{A}_0^T & \mathbf{A}_0^T & \mathbf{A}_0^T \end{bmatrix}\right) h^T \times \begin{bmatrix} \mathbf{A}_1 & \mathbf{O}_{6,5} & -\mathbf{A}_1 & \mathbf{O}_{6,5} & \mathbf{O}_{6,5} \\ \mathbf{O}_{6,5} & \mathbf{A}_1 & \mathbf{A}_1 & \mathbf{O}_{6,5} & \mathbf{O}_{6,5} \end{bmatrix} x_{2D}^T \quad (19)$$

where the vector V_0 is:

$$V_0 = \begin{bmatrix} 1 & 1/2 & 1/2 & 1 & 1 & 1 & 1/2 & 1/4 & 1/4 & 1/2 & 1/2 & 1/2 \end{bmatrix}^T \quad (20)$$

Similarly, we have:

$$\mathbf{Y}_1 = \begin{bmatrix} \mathbf{A}_0 & \mathbf{O}_{3,6} \\ -\mathbf{A}_0 & \mathbf{A}_0 \\ \mathbf{A}_0 & \mathbf{O}_{3,6} \end{bmatrix} \cdot diag(V_1) \times diag\left(\begin{bmatrix} \mathbf{A}_0^T & -\mathbf{A}_0^T & \mathbf{A}_0^T \\ \mathbf{O}_{6,3} & \mathbf{A}_0^T & \mathbf{O}_{6,3} \end{bmatrix}\right) h^T \times \begin{bmatrix} \mathbf{O}_{6,5} & \mathbf{A}_1 & -\mathbf{A}_1 & \mathbf{O}_{6,5} & \mathbf{O}_{6,5} \\ \mathbf{O}_{6,5} & \mathbf{A}_1 & \mathbf{A}_1 & \mathbf{O}_{6,5} & \mathbf{O}_{6,5} \end{bmatrix} x_{2D}^T \quad (21)$$

$$V_1 = \begin{bmatrix} 1/2 & 1/4 & 1/4 & 1/2 & 1/2 & 1/2 & 1 & 1/2 & 1/2 & 1 & 1 & 1 \end{bmatrix}^T \quad (22)$$

$$\mathbf{Y}_2 = \begin{bmatrix} \mathbf{O}_{3,6} & \mathbf{O}_{3,6} \\ \mathbf{A}_0 & \mathbf{O}_{3,6} \\ -\mathbf{A}_0 & \mathbf{A}_0 \end{bmatrix} \cdot diag(V_2) \times diag\left(\begin{bmatrix} \mathbf{O}_{6,3} & \mathbf{A}_0^T & -\mathbf{A}_0^T \\ \mathbf{O}_{6,3} & \mathbf{O}_{6,3} & \mathbf{A}_0^T \end{bmatrix} \right) h^T \times \begin{bmatrix} \mathbf{O}_{6,5} & \mathbf{A}_1 & \mathbf{O}_{6,5} & -\mathbf{A}_1 & \mathbf{O}_{6,5} \\ \mathbf{O}_{6,5} & -\mathbf{A}_1 & -\mathbf{A}_1 & \mathbf{A}_1 & \mathbf{A}_1 \end{bmatrix} x_{2D} \quad (23)$$

$$V_2 = \begin{bmatrix} 1 & 1/2 & 1/2 & 1 & 1 & 1 & 1 & 1/2 & 1/2 & 1 & 1 & 1 \end{bmatrix}^T \quad (24)$$

In the above matrix relations, the 3x6 and 6x5 block matrices are:

$$\mathbf{A}_0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{A}_1 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 1 \end{bmatrix} \quad (25)$$

and $\mathbf{O}_{3,6}$, $\mathbf{O}_{6,3}$, $\mathbf{O}_{6,5}$ are zero matrices of size 3×6, 6×3, and 6×5, respectively. Summing up the obtained partial results $\mathbf{Y}_0$, $\mathbf{Y}_1$ and $\mathbf{Y}_2$, the following output vector $\mathbf{Y}$ containing 25 samples of the filtered image is obtained:

$$\mathbf{Y} = \mathbf{Y}_0 + \mathbf{Y}_1 + \mathbf{Y}_2 = \begin{bmatrix} Y_{00} & Y_{01} & Y_{02} & Y_{10} & Y_{11} & Y_{12} & Y_{20} & Y_{20} & Y_{20} \end{bmatrix}^T \quad (26)$$

Also, we have:

$$h = \begin{bmatrix} h_{00} & h_{01} & h_{02} & h_{10} & h_{11} & h_{12} & h_{20} & h_{21} & h_{22} \end{bmatrix} \quad (27)$$

$$x_{2D} = \begin{bmatrix} x_{00} & x_{01} & x_{02} & x_{03} & x_{04} & x_{10} & x_{11} & \cdots & x_{33} & x_{34} & x_{40} & x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix} \quad (28)$$

The above algorithm for a 2D FIR filtering operation has been derived using a subexpression sharing technique in order to reduce the number of operations due to the fact that there are redundant operations in the direct implementation using 2D convolution. Thus, in direct 2D convolutions there are input data blocks that are overlapping, and avoiding these redundant operations we can obtain important savings in the number of arithmetic operations. Using a subexpression sharing technique, the above algorithm resulted for the 2D FIR filtering operation.

In order to extend 2D filtering operations to a 27x27 kernel we will use next a polyphase decomposition with a decimation factor of 3 and thus we obtain 9 submatrices. For the input matrix we have used a decimation factor of 5 and we have obtained 25 submatrices for a 45x45 input matrix.

The equation (27) can now be reformulated for a polyphase implementation by replacing the vector *h* with the vector *H* defined as below:

$$H = \begin{bmatrix} H_{00} & H_{01} & H_{02} & H_{10} & H_{11} & H_{12} & H_{20} & H_{21} & H_{22} \end{bmatrix}^T \quad (29)$$

The vectors H_{00} ,..., H_{22} for the size 27x27 of the kernel matrix are defined as:

$$H_{00} = \begin{bmatrix} H_{00}^0 & H_{00}^1 & H_{00}^2 & H_{00}^3 & H_{00}^4 & H_{00}^5 & H_{00}^6 & H_{00}^7 & H_{00}^8 \end{bmatrix} \quad (30)$$

where the components of vector H_{00} are given in the group of equations (31):

$$\begin{aligned}
H_{00}^{0} &= \begin{bmatrix} h_{0,0} & h_{0,3} & h_{0,6} & h_{0,9} & h_{0,12} & h_{0,15} & h_{0,18} & h_{0,21} & h_{0,24} \end{bmatrix} \\
H_{00}^{1} &= \begin{bmatrix} h_{3,0} & h_{3,3} & h_{3,6} & h_{3,9} & h_{3,12} & h_{3,15} & h_{3,18} & h_{3,21} & h_{3,24} \end{bmatrix} \\
H_{00}^{2} &= \begin{bmatrix} h_{6,0} & h_{6,3} & h_{6,6} & h_{6,9} & h_{6,12} & h_{6,15} & h_{6,18} & h_{6,21} & h_{6,24} \end{bmatrix} \\
H_{00}^{3} &= \begin{bmatrix} h_{9,0} & h_{9,3} & h_{9,6} & h_{9,9} & h_{9,12} & h_{9,15} & h_{9,18} & h_{9,21} & h_{9,24} \end{bmatrix} \\
H_{00}^{4} &= \begin{bmatrix} h_{12,0} & h_{12,3} & h_{12,6} & h_{12,9} & h_{12,12} & h_{12,15} & h_{12,18} & h_{12,21} & h_{12,24} \end{bmatrix} \\
H_{00}^{5} &= \begin{bmatrix} h_{15,0} & h_{15,3} & h_{15,6} & h_{15,9} & h_{15,12} & h_{15,15} & h_{15,18} & h_{15,21} & h_{15,24} \end{bmatrix} \\
H_{00}^{6} &= \begin{bmatrix} h_{18,0} & h_{18,3} & h_{18,6} & h_{18,9} & h_{18,12} & h_{18,15} & h_{18,18} & h_{18,21} & h_{18,24} \end{bmatrix} \\
H_{00}^{7} &= \begin{bmatrix} h_{21,0} & h_{21,3} & h_{21,6} & h_{21,9} & h_{21,12} & h_{21,15} & h_{21,18} & h_{21,21} & h_{21,24} \end{bmatrix} \\
H_{00}^{8} &= \begin{bmatrix} h_{24,0} & h_{24,3} & h_{24,6} & h_{24,9} & h_{24,12} & h_{24,15} & h_{24,18} & h_{24,21} & h_{24,24} \end{bmatrix}
\end{aligned} \tag{31}$$

Similarly, the other vectors and their components are displayed in the following groups of expressions:

$$H_{01} = \begin{bmatrix} H_{01}^{0} & H_{01}^{1} & H_{01}^{2} & H_{01}^{3} & H_{01}^{4} & H_{01}^{5} & H_{01}^{6} & H_{01}^{7} & H_{01}^{8} \end{bmatrix} \tag{32}$$

where:

$$\begin{aligned}
H_{01}^{0} &= \begin{bmatrix} h_{0,1} & h_{0,4} & h_{0,7} & h_{0,10} & h_{0,13} & h_{0,16} & h_{0,19} & h_{0,22} & h_{0,25} \end{bmatrix} \\
H_{01}^{1} &= \begin{bmatrix} h_{3,1} & h_{3,4} & h_{3,7} & h_{3,10} & h_{3,13} & h_{3,16} & h_{3,19} & h_{3,22} & h_{3,25} \end{bmatrix} \\
H_{01}^{2} &= \begin{bmatrix} h_{6,1} & h_{6,4} & h_{6,7} & h_{6,10} & h_{6,13} & h_{6,16} & h_{6,19} & h_{6,22} & h_{6,25} \end{bmatrix} \\
H_{01}^{3} &= \begin{bmatrix} h_{9,1} & h_{9,4} & h_{9,7} & h_{9,10} & h_{9,13} & h_{9,16} & h_{9,19} & h_{9,22} & h_{9,25} \end{bmatrix} \\
H_{01}^{4} &= \begin{bmatrix} h_{12,1} & h_{12,4} & h_{12,7} & h_{12,10} & h_{12,13} & h_{12,16} & h_{12,19} & h_{12,22} & h_{12,25} \end{bmatrix} \\
H_{01}^{5} &= \begin{bmatrix} h_{15,1} & h_{15,4} & h_{15,7} & h_{15,10} & h_{15,13} & h_{15,16} & h_{15,19} & h_{15,22} & h_{15,25} \end{bmatrix} \\
H_{01}^{6} &= \begin{bmatrix} h_{18,1} & h_{18,4} & h_{18,7} & h_{18,10} & h_{18,13} & h_{18,16} & h_{18,19} & h_{18,22} & h_{18,25} \end{bmatrix} \\
H_{01}^{7} &= \begin{bmatrix} h_{21,1} & h_{21,4} & h_{21,7} & h_{21,10} & h_{21,13} & h_{21,16} & h_{21,19} & h_{21,22} & h_{21,25} \end{bmatrix} \\
H_{01}^{8} &= \begin{bmatrix} h_{24,1} & h_{24,4} & h_{24,7} & h_{24,10} & h_{24,13} & h_{24,16} & h_{24,19} & h_{24,22} & h_{24,25} \end{bmatrix}
\end{aligned} \tag{33}$$

also

$$H_{02} = \begin{bmatrix} H_{02}^{0} & H_{02}^{1} & H_{02}^{2} & H_{02}^{3} & H_{02}^{4} & H_{02}^{5} & H_{02}^{6} & H_{02}^{7} & H_{02}^{8} \end{bmatrix} \tag{34}$$

where:

$$\begin{aligned}
H_{02}^{0} &= \begin{bmatrix} h_{0,2} & h_{0,5} & h_{0,8} & h_{0,11} & h_{0,14} & h_{0,17} & h_{0,20} & h_{0,23} & h_{0,26} \end{bmatrix} \\
H_{02}^{1} &= \begin{bmatrix} h_{3,2} & h_{3,5} & h_{3,8} & h_{3,11} & h_{3,14} & h_{3,17} & h_{3,20} & h_{3,23} & h_{3,26} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
H_{02}^{2} &= \begin{bmatrix} h_{6,2} & h_{6,5} & h_{6,8} & h_{6,11} & h_{6,14} & h_{6,17} & h_{6,20} & h_{6,23} & h_{6,26} \end{bmatrix}\\
H_{02}^{3} &= \begin{bmatrix} h_{9,2} & h_{9,5} & h_{9,8} & h_{9,11} & h_{9,14} & h_{9,17} & h_{9,20} & h_{9,23} & h_{9,26} \end{bmatrix}\\
H_{02}^{4} &= \begin{bmatrix} h_{12,2} & h_{12,5} & h_{12,8} & h_{12,11} & h_{12,14} & h_{12,17} & h_{12,20} & h_{12,23} & h_{12,26} \end{bmatrix}\\
H_{02}^{5} &= \begin{bmatrix} h_{15,2} & h_{15,5} & h_{15,8} & h_{15,11} & h_{15,14} & h_{15,17} & h_{15,20} & h_{15,23} & h_{15,26} \end{bmatrix}\\
H_{02}^{6} &= \begin{bmatrix} h_{18,2} & h_{18,5} & h_{18,8} & h_{18,11} & h_{18,14} & h_{18,17} & h_{18,20} & h_{18,23} & h_{18,26} \end{bmatrix}\\
H_{02}^{7} &= \begin{bmatrix} h_{21,2} & h_{21,5} & h_{21,8} & h_{21,11} & h_{21,14} & h_{21,17} & h_{21,20} & h_{21,23} & h_{21,26} \end{bmatrix}\\
H_{02}^{8} &= \begin{bmatrix} h_{24,2} & h_{24,5} & h_{24,8} & h_{24,11} & h_{24,14} & h_{24,17} & h_{24,20} & h_{24,23} & h_{24,26} \end{bmatrix}
\end{aligned} \tag{35}$$

also

$$H_{10} = \begin{bmatrix} H_{10}^{0} & H_{10}^{1} & H_{10}^{2} & H_{10}^{3} & H_{10}^{4} & H_{10}^{5} & H_{10}^{6} & H_{10}^{7} & H_{10}^{8} \end{bmatrix}$$

where:

$$\begin{aligned}
H_{10}^{0} &= \begin{bmatrix} h_{1,0} & h_{1,3} & h_{1,6} & h_{1,9} & h_{1,12} & h_{1,15} & h_{1,18} & h_{1,21} & h_{1,24} \end{bmatrix}\\
H_{10}^{1} &= \begin{bmatrix} h_{4,0} & h_{4,3} & h_{4,6} & h_{4,9} & h_{4,12} & h_{4,15} & h_{4,18} & h_{4,21} & h_{4,24} \end{bmatrix}\\
H_{10}^{2} &= \begin{bmatrix} h_{7,0} & h_{7,3} & h_{7,6} & h_{7,9} & h_{7,12} & h_{7,15} & h_{7,18} & h_{7,21} & h_{7,24} \end{bmatrix}\\
H_{10}^{3} &= \begin{bmatrix} h_{10,0} & h_{10,3} & h_{10,6} & h_{10,9} & h_{10,12} & h_{10,15} & h_{10,18} & h_{10,21} & h_{10,24} \end{bmatrix}\\
H_{10}^{4} &= \begin{bmatrix} h_{13,0} & h_{13,3} & h_{13,6} & h_{13,9} & h_{13,12} & h_{13,15} & h_{13,18} & h_{13,21} & h_{13,24} \end{bmatrix}\\
H_{10}^{5} &= \begin{bmatrix} h_{16,0} & h_{16,3} & h_{16,6} & h_{16,9} & h_{16,12} & h_{16,15} & h_{16,18} & h_{16,21} & h_{16,24} \end{bmatrix}\\
H_{10}^{6} &= \begin{bmatrix} h_{19,0} & h_{19,3} & h_{19,6} & h_{19,9} & h_{19,12} & h_{19,15} & h_{19,18} & h_{19,21} & h_{19,24} \end{bmatrix}\\
H_{10}^{7} &= \begin{bmatrix} h_{22,0} & h_{22,3} & h_{22,6} & h_{22,9} & h_{22,12} & h_{22,15} & h_{22,18} & h_{22,21} & h_{22,24} \end{bmatrix}\\
H_{10}^{8} &= \begin{bmatrix} h_{25,0} & h_{25,3} & h_{25,6} & h_{25,9} & h_{25,12} & h_{25,15} & h_{25,18} & h_{25,21} & h_{25,24} \end{bmatrix}
\end{aligned} \tag{36}$$

and

$$H_{11} = \begin{bmatrix} H_{11}^{0} & H_{11}^{1} & H_{11}^{2} & H_{11}^{3} & H_{11}^{4} & H_{11}^{5} & H_{11}^{6} & H_{11}^{7} & H_{11}^{8} \end{bmatrix}$$

where:

$$\begin{aligned}
H_{11}^{0} &= \begin{bmatrix} h_{1,1} & h_{1,4} & h_{1,7} & h_{1,10} & h_{1,13} & h_{1,16} & h_{1,19} & h_{1,22} & h_{1,25} \end{bmatrix}\\
H_{11}^{1} &= \begin{bmatrix} h_{4,1} & h_{4,4} & h_{4,7} & h_{4,10} & h_{4,13} & h_{4,16} & h_{4,19} & h_{4,22} & h_{4,25} \end{bmatrix}\\
H_{11}^{2} &= \begin{bmatrix} h_{7,1} & h_{7,4} & h_{7,7} & h_{7,10} & h_{7,13} & h_{7,16} & h_{7,19} & h_{7,22} & h_{7,25} \end{bmatrix}\\
H_{11}^{3} &= \begin{bmatrix} h_{10,1} & h_{10,4} & h_{10,7} & h_{10,10} & h_{10,13} & h_{10,16} & h_{10,19} & h_{10,22} & h_{10,25} \end{bmatrix}\\
H_{11}^{4} &= \begin{bmatrix} h_{13,1} & h_{13,4} & h_{13,7} & h_{13,10} & h_{13,13} & h_{13,16} & h_{13,19} & h_{13,22} & h_{13,25} \end{bmatrix}\\
H_{11}^{5} &= \begin{bmatrix} h_{16,1} & h_{16,4} & h_{16,7} & h_{16,10} & h_{16,13} & h_{16,16} & h_{16,19} & h_{16,22} & h_{16,25} \end{bmatrix}
\end{aligned} \tag{37}$$

$$H_{11}^{6} = \begin{bmatrix} h_{19,1} & h_{19,4} & h_{19,7} & h_{19,10} & h_{19,13} & h_{19,16} & h_{19,19} & h_{19,22} & h_{19,25} \end{bmatrix}$$
$$H_{11}^{7} = \begin{bmatrix} h_{22,1} & h_{22,4} & h_{22,7} & h_{22,10} & h_{22,13} & h_{22,16} & h_{22,19} & h_{22,22} & h_{22,25} \end{bmatrix}$$
$$H_{11}^{8} = \begin{bmatrix} h_{25,1} & h_{25,4} & h_{25,7} & h_{25,10} & h_{25,13} & h_{25,16} & h_{25,19} & h_{25,22} & h_{25,25} \end{bmatrix}$$

and

$$H_{12} = \begin{bmatrix} H_{12}^{0} & H_{12}^{1} & H_{12}^{2} & H_{12}^{3} & H_{12}^{4} & H_{12}^{5} & H_{12}^{6} & H_{12}^{7} & H_{12}^{8} \end{bmatrix} \tag{38}$$

where:

$$\begin{aligned}
H_{12}^{0} &= \begin{bmatrix} h_{1,2} & h_{1,5} & h_{1,8} & h_{1,11} & h_{1,14} & h_{1,17} & h_{1,20} & h_{1,23} & h_{1,26} \end{bmatrix}\\
H_{12}^{1} &= \begin{bmatrix} h_{4,2} & h_{4,5} & h_{4,8} & h_{4,11} & h_{4,14} & h_{4,17} & h_{4,20} & h_{4,23} & h_{4,26} \end{bmatrix}\\
H_{12}^{2} &= \begin{bmatrix} h_{7,2} & h_{7,5} & h_{7,8} & h_{7,11} & h_{7,14} & h_{7,17} & h_{7,20} & h_{7,23} & h_{7,26} \end{bmatrix}\\
H_{12}^{3} &= \begin{bmatrix} h_{10,2} & h_{10,5} & h_{10,8} & h_{10,11} & h_{10,14} & h_{10,17} & h_{10,20} & h_{10,23} & h_{10,26} \end{bmatrix}\\
H_{12}^{4} &= \begin{bmatrix} h_{13,2} & h_{13,5} & h_{13,8} & h_{13,11} & h_{13,14} & h_{13,17} & h_{13,20} & h_{13,23} & h_{13,26} \end{bmatrix}\\
H_{12}^{5} &= \begin{bmatrix} h_{16,2} & h_{16,5} & h_{16,8} & h_{16,11} & h_{16,14} & h_{16,17} & h_{16,20} & h_{16,23} & h_{16,26} \end{bmatrix}\\
H_{12}^{6} &= \begin{bmatrix} h_{19,2} & h_{19,5} & h_{19,8} & h_{19,11} & h_{19,14} & h_{19,17} & h_{19,20} & h_{19,23} & h_{19,26} \end{bmatrix}\\
H_{12}^{7} &= \begin{bmatrix} h_{22,2} & h_{22,5} & h_{22,8} & h_{22,11} & h_{22,14} & h_{22,17} & h_{22,20} & h_{22,23} & h_{22,26} \end{bmatrix}\\
H_{12}^{8} &= \begin{bmatrix} h_{25,2} & h_{25,5} & h_{25,8} & h_{25,11} & h_{25,14} & h_{25,17} & h_{25,20} & h_{25,23} & h_{25,26} \end{bmatrix}
\end{aligned} \tag{39}$$

and also

$$H_{20} = \begin{bmatrix} H_{20}^{0} & H_{20}^{1} & H_{20}^{2} & H_{20}^{3} & H_{20}^{4} & H_{20}^{5} & H_{20}^{6} & H_{20}^{7} & H_{20}^{8} \end{bmatrix} \tag{40}$$

where:

$$\begin{aligned}
H_{20}^{0} &= \begin{bmatrix} h_{2,0} & h_{2,3} & h_{2,6} & h_{2,9} & h_{2,12} & h_{2,15} & h_{2,18} & h_{2,21} & h_{2,24} \end{bmatrix}\\
H_{20}^{1} &= \begin{bmatrix} h_{5,0} & h_{5,3} & h_{5,6} & h_{5,9} & h_{5,12} & h_{5,15} & h_{5,18} & h_{5,21} & h_{5,24} \end{bmatrix}\\
H_{20}^{2} &= \begin{bmatrix} h_{8,0} & h_{8,3} & h_{8,6} & h_{8,9} & h_{8,12} & h_{8,15} & h_{8,18} & h_{8,21} & h_{8,24} \end{bmatrix}\\
H_{20}^{3} &= \begin{bmatrix} h_{11,0} & h_{11,3} & h_{11,6} & h_{11,9} & h_{11,12} & h_{11,15} & h_{11,18} & h_{11,21} & h_{11,24} \end{bmatrix}\\
H_{20}^{4} &= \begin{bmatrix} h_{14,0} & h_{14,3} & h_{14,6} & h_{14,9} & h_{14,12} & h_{14,15} & h_{14,18} & h_{14,21} & h_{14,24} \end{bmatrix}\\
H_{20}^{5} &= \begin{bmatrix} h_{17,0} & h_{17,3} & h_{17,6} & h_{17,9} & h_{17,12} & h_{17,15} & h_{17,18} & h_{17,21} & h_{17,24} \end{bmatrix}\\
H_{20}^{6} &= \begin{bmatrix} h_{20,0} & h_{20,3} & h_{20,6} & h_{20,9} & h_{20,12} & h_{20,15} & h_{20,18} & h_{20,21} & h_{20,24} \end{bmatrix}\\
H_{20}^{7} &= \begin{bmatrix} h_{23,0} & h_{23,3} & h_{23,6} & h_{23,9} & h_{23,12} & h_{23,15} & h_{23,18} & h_{23,21} & h_{23,24} \end{bmatrix}\\
H_{20}^{8} &= \begin{bmatrix} h_{26,0} & h_{26,3} & h_{26,6} & h_{26,9} & h_{26,12} & h_{26,15} & h_{26,18} & h_{26,21} & h_{26,24} \end{bmatrix}
\end{aligned} \tag{41}$$

and

$$H_{21} = \begin{bmatrix} H_{21}^{0} & H_{21}^{1} & H_{21}^{2} & H_{21}^{3} & H_{21}^{4} & H_{21}^{5} & H_{21}^{6} & H_{21}^{7} & H_{21}^{8} \end{bmatrix} \tag{42}$$

where:

$$\begin{gathered}
H_{21}^0 = \begin{bmatrix} h_{2,1} & h_{2,4} & h_{2,7} & h_{2,10} & h_{2,13} & h_{2,16} & h_{2,19} & h_{2,22} & h_{2,25} \end{bmatrix} \\
H_{21}^1 = \begin{bmatrix} h_{5,1} & h_{5,4} & h_{5,7} & h_{5,10} & h_{5,13} & h_{5,16} & h_{5,19} & h_{5,22} & h_{5,25} \end{bmatrix} \\
H_{21}^2 = \begin{bmatrix} h_{8,1} & h_{8,4} & h_{8,7} & h_{8,10} & h_{8,13} & h_{8,16} & h_{8,19} & h_{8,22} & h_{8,25} \end{bmatrix} \\
H_{21}^3 = \begin{bmatrix} h_{11,1} & h_{11,4} & h_{11,7} & h_{11,10} & h_{11,13} & h_{11,16} & h_{11,19} & h_{11,22} & h_{11,25} \end{bmatrix} \\
H_{21}^4 = \begin{bmatrix} h_{14,1} & h_{14,4} & h_{14,7} & h_{14,10} & h_{14,13} & h_{14,16} & h_{14,19} & h_{14,22} & h_{14,25} \end{bmatrix} \\
H_{21}^5 = \begin{bmatrix} h_{17,1} & h_{17,4} & h_{17,7} & h_{17,10} & h_{17,13} & h_{17,16} & h_{17,19} & h_{17,22} & h_{17,25} \end{bmatrix} \\
H_{21}^6 = \begin{bmatrix} h_{20,1} & h_{20,4} & h_{20,7} & h_{20,10} & h_{20,13} & h_{20,16} & h_{20,19} & h_{20,22} & h_{20,25} \end{bmatrix} \\
H_{21}^7 = \begin{bmatrix} h_{23,1} & h_{23,4} & h_{23,7} & h_{23,10} & h_{23,13} & h_{23,16} & h_{23,19} & h_{23,22} & h_{23,25} \end{bmatrix} \\
H_{21}^8 = \begin{bmatrix} h_{26,1} & h_{26,4} & h_{26,7} & h_{26,10} & h_{26,13} & h_{26,16} & h_{26,19} & h_{26,22} & h_{26,25} \end{bmatrix}
\end{gathered} \tag{43}$$

and

$$H_{22} = \begin{bmatrix} H_{22}^0 & H_{22}^1 & H_{22}^2 & H_{22}^3 & H_{22}^4 & H_{22}^5 & H_{22}^6 & H_{22}^7 & H_{22}^8 \end{bmatrix} \tag{44}$$

where:

$$\begin{gathered}
H_{22}^0 = \begin{bmatrix} h_{2,2} & h_{2,5} & h_{2,8} & h_{2,11} & h_{2,14} & h_{2,17} & h_{2,20} & h_{2,23} & h_{2,26} \end{bmatrix} \\
H_{22}^1 = \begin{bmatrix} h_{5,2} & h_{5,5} & h_{5,8} & h_{5,11} & h_{5,14} & h_{5,17} & h_{5,20} & h_{5,23} & h_{5,26} \end{bmatrix} \\
H_{22}^2 = \begin{bmatrix} h_{8,2} & h_{8,5} & h_{8,8} & h_{8,11} & h_{8,14} & h_{8,17} & h_{8,20} & h_{8,23} & h_{8,26} \end{bmatrix} \\
H_{22}^3 = \begin{bmatrix} h_{11,2} & h_{11,5} & h_{11,8} & h_{11,11} & h_{11,14} & h_{11,17} & h_{11,20} & h_{11,23} & h_{11,26} \end{bmatrix} \\
H_{22}^4 = \begin{bmatrix} h_{14,2} & h_{14,5} & h_{14,8} & h_{14,11} & h_{14,14} & h_{14,17} & h_{14,20} & h_{14,23} & h_{14,26} \end{bmatrix} \\
H_{22}^5 = \begin{bmatrix} h_{17,2} & h_{17,5} & h_{17,8} & h_{17,11} & h_{17,14} & h_{17,17} & h_{17,20} & h_{17,23} & h_{17,26} \end{bmatrix} \\
H_{22}^6 = \begin{bmatrix} h_{20,2} & h_{20,5} & h_{20,8} & h_{20,11} & h_{20,14} & h_{20,17} & h_{20,20} & h_{20,23} & h_{20,26} \end{bmatrix} \\
H_{22}^7 = \begin{bmatrix} h_{23,2} & h_{23,5} & h_{23,8} & h_{23,11} & h_{23,14} & h_{23,17} & h_{23,20} & h_{23,23} & h_{23,26} \end{bmatrix} \\
H_{22}^8 = \begin{bmatrix} h_{26,2} & h_{26,5} & h_{26,8} & h_{26,11} & h_{26,14} & h_{26,17} & h_{26,20} & h_{26,23} & h_{26,26} \end{bmatrix}
\end{gathered} \tag{45}$$

For the input matrix of size 45x45 we have organized the data as shown below. We also replace the vector x_{2D} with the vector X_{2D} defined as follows:

$$X_{2D} = \begin{bmatrix} X_{00} & X_{01} & X_{02} & X_{03} & X_{04} & X_{10} & X_{11} & \cdots & X_{33} & X_{34} & X_{40} & X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}^T \tag{46}$$

The vectors X_{00}, X_{01} ..., X_{44} are defined below:

$$X_{00} = \begin{bmatrix} X_{00}^8 & X_{00}^7 & X_{00}^6 & X_{00}^5 & X_{00}^4 & X_{00}^3 & X_{00}^2 & X_{00}^1 & X_{00}^0 \end{bmatrix} \tag{47}$$

where $X_{00}^0,..,X_{00}^8$ are derived from the following vectors $Xa_{00}^0,..,Xa_{00}^0$, whose elements are taken in reversed order:

$$
\begin{aligned}
Xa_{00}^{0} &= \begin{bmatrix} x_{0,0} & x_{0,5} & x_{0,10} & x_{0,15} & x_{0,20} & x_{0,25} & x_{0,30} & x_{0,35} & x_{0,40} \end{bmatrix} \\
Xa_{00}^{1} &= \begin{bmatrix} x_{5,0} & x_{5,5} & x_{5,10} & x_{5,15} & x_{5,20} & x_{5,25} & x_{5,30} & x_{5,35} & x_{5,40} \end{bmatrix} \\
Xa_{00}^{2} &= \begin{bmatrix} x_{10,0} & x_{10,5} & x_{10,10} & x_{10,15} & x_{10,20} & x_{10,25} & x_{10,30} & x_{10,35} & x_{10,40} \end{bmatrix} \\
Xa_{00}^{3} &= \begin{bmatrix} x_{15,0} & x_{15,5} & x_{15,10} & x_{15,15} & x_{15,20} & x_{15,25} & x_{15,30} & x_{15,35} & x_{15,40} \end{bmatrix} \\
Xa_{00}^{4} &= \begin{bmatrix} x_{20,0} & x_{20,5} & x_{20,10} & x_{20,15} & x_{20,20} & x_{20,25} & x_{20,30} & x_{20,35} & x_{20,40} \end{bmatrix} \\
Xa_{00}^{5} &= \begin{bmatrix} x_{25,0} & x_{25,5} & x_{25,10} & x_{25,15} & x_{25,20} & x_{25,25} & x_{25,30} & x_{25,35} & x_{25,40} \end{bmatrix} \\
Xa_{00}^{6} &= \begin{bmatrix} x_{30,0} & x_{30,5} & x_{30,10} & x_{30,15} & x_{30,20} & x_{30,25} & x_{30,30} & x_{30,35} & x_{30,40} \end{bmatrix} \\
Xa_{00}^{7} &= \begin{bmatrix} x_{35,0} & x_{35,5} & x_{35,10} & x_{35,15} & x_{35,20} & x_{35,25} & x_{35,30} & x_{35,35} & x_{35,40} \end{bmatrix} \\
Xa_{00}^{8} &= \begin{bmatrix} x_{40,0} & x_{40,5} & x_{40,10} & x_{40,15} & x_{40,20} & x_{40,25} & x_{40,30} & x_{40,35} & x_{40,40} \end{bmatrix}
\end{aligned} \tag{48}
$$

Similarly, we have:

$$X_{01} = \begin{bmatrix} X_{01}^{8} & X_{01}^{7} & X_{01}^{6} & X_{01}^{5} & X_{01}^{4} & X_{01}^{3} & X_{01}^{2} & X_{01}^{1} & X_{01}^{0} \end{bmatrix} \tag{49}$$

with

$$
\begin{aligned}
Xa_{01}^{0} &= \begin{bmatrix} x_{0,1} & x_{0,6} & x_{0,11} & x_{0,16} & x_{0,21} & x_{0,26} & x_{0,31} & x_{0,36} & x_{0,41} \end{bmatrix} \\
Xa_{01}^{1} &= \begin{bmatrix} x_{5,1} & x_{5,6} & x_{5,11} & x_{5,16} & x_{5,21} & x_{5,26} & x_{5,31} & x_{5,36} & x_{5,41} \end{bmatrix} \\
Xa_{01}^{2} &= \begin{bmatrix} x_{10,1} & x_{10,6} & x_{10,11} & x_{10,16} & x_{10,21} & x_{10,26} & x_{10,31} & x_{10,36} & x_{10,41} \end{bmatrix} \\
Xa_{01}^{3} &= \begin{bmatrix} x_{15,1} & x_{15,6} & x_{15,11} & x_{15,16} & x_{15,21} & x_{15,26} & x_{15,31} & x_{15,36} & x_{15,41} \end{bmatrix} \\
Xa_{01}^{4} &= \begin{bmatrix} x_{20,1} & x_{20,6} & x_{20,11} & x_{20,16} & x_{20,21} & x_{20,26} & x_{20,31} & x_{20,36} & x_{20,41} \end{bmatrix} \\
Xa_{01}^{5} &= \begin{bmatrix} x_{25,1} & x_{25,6} & x_{25,11} & x_{25,16} & x_{25,21} & x_{25,26} & x_{25,31} & x_{25,36} & x_{25,41} \end{bmatrix} \\
Xa_{01}^{6} &= \begin{bmatrix} x_{30,1} & x_{30,6} & x_{30,11} & x_{30,16} & x_{30,21} & x_{30,26} & x_{30,31} & x_{30,36} & x_{30,41} \end{bmatrix} \\
Xa_{01}^{7} &= \begin{bmatrix} x_{35,1} & x_{35,6} & x_{35,11} & x_{35,16} & x_{35,21} & x_{35,26} & x_{35,31} & x_{35,36} & x_{35,41} \end{bmatrix} \\
Xa_{01}^{8} &= \begin{bmatrix} x_{40,1} & x_{40,6} & x_{40,11} & x_{40,16} & x_{40,21} & x_{40,26} & x_{40,31} & x_{40,36} & x_{40,41} \end{bmatrix}
\end{aligned} \tag{50}
$$

and so on, until the last vector:

$$X_{44} = \begin{bmatrix} X_{44}^{8} & X_{44}^{7} & X_{44}^{6} & X_{44}^{5} & X_{44}^{4} & X_{44}^{3} & X_{44}^{2} & X_{44}^{1} & X_{44}^{0} \end{bmatrix} \tag{51}$$

where

$$
\begin{aligned}
Xa_{44}^{0} &= \begin{bmatrix} x_{4,4} & x_{4,9} & x_{4,14} & x_{4,19} & x_{4,24} & x_{4,29} & x_{4,34} & x_{4,39} & x_{4,44} \end{bmatrix} \\
Xa_{44}^{1} &= \begin{bmatrix} x_{9,4} & x_{9,9} & x_{9,14} & x_{9,19} & x_{9,24} & x_{9,29} & x_{9,34} & x_{9,39} & x_{9,44} \end{bmatrix} \\
Xa_{44}^{2} &= \begin{bmatrix} x_{14,4} & x_{14,9} & x_{14,14} & x_{14,19} & x_{14,24} & x_{14,29} & x_{14,34} & x_{14,39} & x_{14,44} \end{bmatrix} \\
Xa_{44}^{3} &= \begin{bmatrix} x_{19,4} & x_{19,9} & x_{19,14} & x_{19,19} & x_{19,24} & x_{19,29} & x_{19,34} & x_{19,39} & x_{19,44} \end{bmatrix}
\end{aligned}
$$

$$\begin{aligned}
Xa_{44}^{4} &= \begin{bmatrix} x_{24,4} & x_{24,9} & x_{24,14} & x_{24,19} & x_{24,24} & x_{24,29} & x_{24,34} & x_{24,39} & x_{24,44} \end{bmatrix} \\
Xa_{44}^{5} &= \begin{bmatrix} x_{29,4} & x_{29,9} & x_{29,14} & x_{29,19} & x_{29,24} & x_{29,29} & x_{29,34} & x_{29,39} & x_{29,44} \end{bmatrix} \\
Xa_{44}^{6} &= \begin{bmatrix} x_{34,4} & x_{34,9} & x_{34,14} & x_{34,19} & x_{34,24} & x_{34,29} & x_{34,34} & x_{34,39} & x_{34,44} \end{bmatrix} \\
Xa_{44}^{7} &= \begin{bmatrix} x_{39,4} & x_{39,9} & x_{39,14} & x_{39,19} & x_{39,24} & x_{39,29} & x_{39,34} & x_{39,39} & x_{39,44} \end{bmatrix} \\
Xa_{44}^{8} &= \begin{bmatrix} x_{44,4} & x_{44,9} & x_{44,14} & x_{44,19} & x_{44,24} & x_{44,29} & x_{44,34} & x_{44,39} & x_{44,44} \end{bmatrix}
\end{aligned} \tag{52}$$

Due to the lack of space, we have shown only the first two elements and the last one. Applying the decimation with 5 instead of the input matrix of 45x45 we obtain 25 matrices of size 9x9 that have been reformulated as 25 vectors with 81 elements. Thus, we have decomposed a 2D FIR filtering operation with a 27x27 kernel and a 45x45 input matrix into 25 inner product operations.

Conclusions

The analytical design procedure described above is simple and efficient, leading to 2D filters of accurate shape. The prototype chosen here is a wide-band low-pass filter with a specified bandwidth, to which a simple frequency mapping is apllied, yielding directly the factored frequency response of the desired 2D circular filter. The method is more general and can be also applied to obtain 2D filters with other shapes. Also, an efficient implementation structure at system level for 2D FIR filters has been proposed. Employing the polyphase decomposition and the block filtering approach, a given 2D filtering operation of an image with a large size filter kernel can be achieved equivalently as a sum of filtering operations with several smaller size kernels. Instead of the redundant and resource consuming direct convolution of an image with a large kernel (in our case, 27x27), the decimation by given factors and block filtering are used, such that computational complexity of the overall filtering task decreases significantly.

In further work, we will focus on determining the optimal decimation factors for the filter kernel and input image in order to obtain the most efficient structure. Also, we envisage developing such implementation structures for other types of 2D filters, all having a low arithmetic complexity and high parallelism.

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