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DIGITAL IMAGE RESTORATION USING LINEAR PDE-BASED FILTERING MODELS

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Rezumat. *Modelele de filtrare a zgomotului Gaussian bazate pe ecuații cu derivate parțiale (PDE) liniare sunt discutate în această lucrare. Soluțiile de restaurare a imaginilor digitale bazate pe ecuații de difuzie liniare sunt mai întâi descrise. În continuare, propriile noastre contribuții în acest domeniu al procesării de imagini, reprezentând modele PDE liniare de filtrare eficiente bazate pe ecuații diferențiale hiperbolice și stocastice, sunt prezentate. Rezultate ale experimentelor noastre de filrare sunt de asemenea oferite în acest articol.*

Abstract. *The linear partial differential equation (PDE) - based models for Gaussian noise removal are discussed in this paper. The digital image denoising and restoration solutions based on linear diffusion equations are surveyed first. Then, our own contributions in this image processing domain, representing some effective linear PDE-based filtering models based on hyperbolic and stochastic differential equations, are presented. The results of our denoising experiments are also provided in this article.*

Keywords: image denoising and restoration, Gaussian noise, unintended effects, linear diffusion, equation, hyperbolic PDE model, stochastic equation.
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1. Introduction

Partial differential equation (PDE) - based models have been applied successfully in the digital image processing and analysis domain in the last 35 years, because the conventional techniques have many drawbacks and cannot solve properly some important tasks related to this field [1]. Such a task is the preservation of the essential details during the denoising and restoration processes.

Second-order nonlinear PDE-based filtering schemes, like the Perona-Malik anisotropic diffusion model [2], TV-ROF Denoising [3] and other diffusion-based and variational methods [4], outperform the classic 2D filters [1], since they remove the additive Gaussian noise, overcome the blurring effect and preserve the essential features, like the edges and corners. Unfortunately, these nonlinear diffusion-based filters may generate the unintended effects, like the staircasing.

Some improved nonlinear PDE variational models that alleviate this undesired effect, such as the Adaptive TV denoising [5], TV-L1 model [6], anisotropic

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HDTV regularizer [7] and the fourth-order You-Kaveh isotropic diffusion scheme [8], have been introduced in the last decades. Although the improved nonlinear PDE-based methods reduce that staircase effect, they do not remove it completely. Also, they have other disadvantages, like the high computational cost, which means a high running time, and the over-filtering effect.

The linear PDE-based denoising algorithms, which are disscused in this paper, provide an effective image restoration and could address some drawbacks of the nonlinear PDE models, although they have their own disadvantages. The linear diffusion schemes surveyed in the next section have long been considered the simplest PDE–based image filtering approaches. Their main drawbacks are the image blurring effect, which could corrupt the boundaries and other features, and the absence of the localization property. They could also dislocate the edges when moving from finer to coarser scales [9].

We have developed many linear and nonlinear PDE-based filtering techniques that reduces successfully the additive noise, preserve the image details and overcome the undesired effects, in the last 15 years [10-15]. They solve properly the drawbacks of the existing linear and nonlinear PDE-based filters. These improved linear PDE-based denoising solutions, which represent our most important contributions in this image processing field, are disscused in the third section. This research article finalizes with a conclusions' section and a list of references.

2. Linear diffusion-based image filtering models

Diffusion constitutes the physics process resulting from random motion of molecules by which there is a net flow of matter from a region of high concentration to a region of low concentration. It equilibrates the concentration differences without creating or destroying the mass and it is expressed by the following Fick's *first law of diffusion* [16]:

$$J = -D \cdot \nabla u \tag{1}$$

where J is the diffusion flux, D represents the diffusivity and ∇u is the concentration gradient. Since the mass is conserved in this process, one may apply the continuity equation [16], which is a conservation law describing the physical transport: $\frac{\partial u}{\partial t} = -div\, J$. By replacing the flux given by (1) in this equation, it results the diffusion equation:

$$\frac{\partial u}{\partial t} = div(D \cdot \nabla u) \tag{2}$$

Diffusion equation (2), which is also called *heat equation*, is very useful in image processing and analysis, where the concentration is identified to the grayscale value at a given location [4]. Thus, if $u:\Omega \subseteq R^2 \to R$ is a gray-level evolving image, the observation $I \in L^1(R^2)$ is affected by the additive white Gaussian noise (AWGN) according to the formation model $I = Hu + N(0,\sigma^2)$, where H : $\mathfrak{H} \to$K, K a real Hilbert space, is a bounded linear operator. So, the next diffusion model results for this denoising process:

$$\begin{cases} \dfrac{\partial u}{\partial t} = div(D(x,y,t)\cdot \nabla u) \\ \qquad\qquad\qquad\qquad\qquad ,(x,y)\in \Omega \subseteq R^2 \\ u(0,x,y) = I(x,y) \end{cases} \quad (3)$$

If the diffusivity function (diffusion tensor) *D* (*x*, *y*, *t*) does not depend on *u* itself, then (3) is a *linear diffusion* process. Otherwise, if *D* is a function of *u*, then (3) represents a *nonlinear diffusion*. If the tensor is constant over the entire domain $(D(x,y,t)=\alpha, \forall (x,y)\in \Omega)$, then the PDE in (3) is a *homogeneous*, or *isotropic diffusion*. If *D* represents a space-dependent function, then this diffusion process is *inhomogeneous*, or *anisotropic* [4, 13].

Since $\frac{\partial u}{\partial t} = div(D(x,y,t)\cdot \nabla u) = D(x,y,t)\Delta u + \nabla D(x,y,t)\cdot \nabla u$, one obtains the next linear isotropic diffusion model:

$$\begin{cases} \dfrac{\partial u}{\partial t} = div(\alpha \nabla u) = \alpha \cdot \Delta u \\ \qquad\qquad\qquad\qquad\qquad , \ (x,y)\in \Omega, \\ u(0,x,y) = I(x,y) \end{cases} \quad (4)$$

where $\alpha > 0$. The PDE model (4) represents a heat transfer equation that is a parabolic partial differential equation describing the temperature variation distribution [17]. The finite difference-based numerical approximation of (4) provides an iterative algorithm that evolves the noisy observation to the denoised image.

This linear diffusion-based denoising solution is equivalent to the 2D Gaussian filtering [1]. The parabolic PDE in (4) has a unique solution of the form:

$$u(x,y,t) = \begin{cases} I(x,y), & t = 0 \\ (G_\sigma * I)(x,y), & t > 0 \end{cases}, \quad (5)$$

where $\sigma = \sqrt{2t}$ and the two-dimension Gaussian filter kernel $G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$.

This shows that performing a linear diffusion for the time *t* with $\alpha = 1$ equivales to performing the Gaussian denoising with $\sigma = \sqrt{2t}$.

Linear PDE-based filtering methods have important advantages, such as the easyness of handling, fast execution and low running time. Also, this formulation in terms of a diffusion equation is much more natural and posesses a higher generalization potential than the operation of convolution with 2D Gaussians.

Unfortunately, a disadvantage of the linear diffusion approaches is the undesirable blurring effect. The isotropic diffusion performs a homogeneous denoising, blurring equally in an all the directions. These techniques remove properly the Gaussian noise, but may blur some details, since they generate diffusion across the boundaries. These linear PDE denoising schemes may also dislocate the image edges when moving from finer to coarser scales [9]. Also, the linear diffusion model (4) has no localization property, its solution propagating with infinite speed.

Some existing techniques aim to solve these issues by introducing some modifications of the filter kernels or transforming the linear models into nonlinear diffusion schemes. Such a detail-preserving solution is the *directed diffusion* process [18]. This process incorporates a-priori knowledge about the details to be preserved into the linear PDE model. A directed diffusion-based approach was proposed by R. Illner and H. Neunzert [18]. It provides some important background information in the form of an auxiliary image, *A*, as following:

$$\begin{cases} \dfrac{\partial u}{\partial t} = A\Delta u - u\Delta A \\ u(0, x, y) = g(x, y) \end{cases}, \ (x, y) \in \Omega, \tag{6}$$

An important category of linear PDE-based filtering models is that of the *linear complex diffusion schemes*. An effective linear complex diffusion-based image filter proposed in [19] has the form:

$$\begin{cases} \dfrac{\partial u}{\partial t} = u_t = c u_{xx}, t > 0, x \in R \\ u(x;0) = u_0 \in R, \ c, u \in C \end{cases} \tag{7}$$

The partial differential equation (7) represents a generalization of the linear diffusion-based equation in (4) for $c \in R$. The diffusion model is well-posed for $c > 0$ [19]. A linear complex diffusion-based denoising example is displayed in Fig. 1.

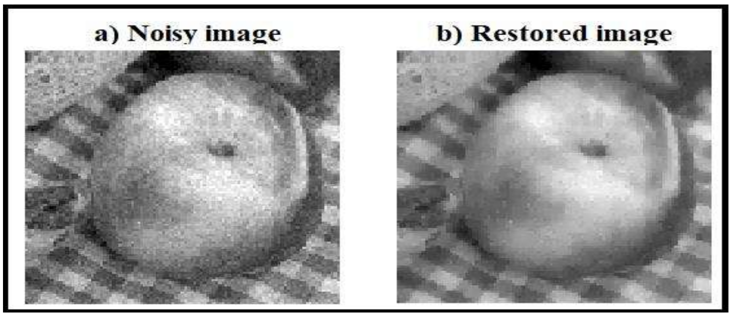

Fig. 1 3.AWGN corrupted image restored by linear complex diffusion

3. Additive noise removal using linear hyperbolic PDE models and stochastic equations

Our main contributions in the linear PDE-based denoising and restoration field are described briefly in this section. They represent improved linear PDE filtering models that solve the limitations of the existing linear PDE-based noise reduction techniques.

Thus, we introduced two restoration algorithms based on linear second-order hyperbolic PDEs with boundary conditions [10, 11]. The first hyperbolic PDE-based model that is disseminated in [10] has the next form:

$$\begin{cases} \lambda \frac{\partial^2 u}{\partial t^2} + \gamma^2 \frac{\partial u}{\partial t} - \alpha \nabla^2 u + \zeta (u - u_0) = 0 \\ u(0, x, y) = u_0(x, y) \qquad , \ (x, y) \in \Omega \\ \frac{\partial u}{\partial t}(0, x, y) = u_1(x, y) \\ u(t, x, y) = 0, \quad \forall t \geq 0, \ (x, y) \in \partial\Omega \end{cases} \tag{8}$$

where image domain $\Omega \subseteq (0, \infty) \times R^2$, $\lambda, \gamma, \alpha \in (0,3]$, $\zeta \in (0,0.5]$ and $u_0 \in L^2(R^2)$ is the observed image corrupted by AWGN.

This second–order PDE model is well-posed, admitting a unique variational solution which is continuous in *t* with values in $L^2(R^2)$. Also, if $u_0 \in H^k(R^2)$, then $u(t) \in H^k(R^2), \forall t \geq 0$ [10]. It also represents a non-Fourier model for the heat propagation, because its unique and weak solution is propagating with finite speed. We also demonstrated that hyperbolic model (8) possesses the localization property [9], unlike others linear PDE-based filtering schemes.

The solution of the proposed linear PDE model is approximated numerically applying the finite-difference method [20]. The next explicit iterative numerical approximation scheme is computed in [10]:

$$u^{n+1}(i,j) = \frac{2\zeta - 4\lambda}{2\lambda + \gamma^2} u^n(i,j) + \frac{\gamma^2 - 2\lambda}{2\lambda + \gamma^2} u^{n-1}(i,j) + 2\zeta u^0(i,j) + \\ + 2\alpha\left(u^n(i+1,j) + u^n(i-1,j) + u^n(i,j+1) + u^n(i,j-1) - 4u^n(i,j)\right) \quad (9)$$

where $u^0(i,j) = u_0(i,j)$. The number of iterations, *N*, is rather low, since this discretization numerical algorithm converges fast to the optimal denoised image.

This technique was successfully tested on hundreds of images corrupted by various amounts of AWGN. It obtained satisfactory filtering results while preserving the image details. Given its hyperbolic character, it even enhances the image edges. It also overcomes other undesirable effects, like staircasing [21] and speckle noise [8]. Some method comparison results illustrating the effectiveness of the described approach are described in Table 1 and Fig. 2 [10].

Table 1. Method comparison: average PSNR values of several filtering techniques

Proposed PDE-based filter	Linear PDE-based filter (heat equation)	Average filter	Wiener 2D	Perona-Malik	TV Denoising
27.1 (dB)	24.3 (dB)	23.2 (dB)	24.5 (dB)	25.8 (dB)	24.3(dB)

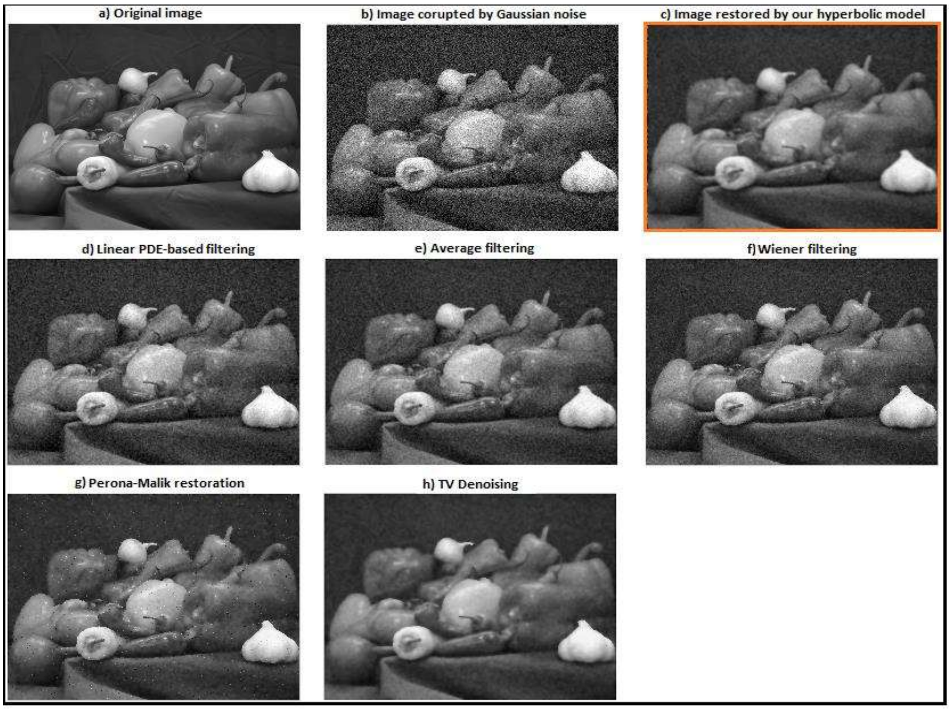

Fig. 2 Method comparison: output restoration of various denoising methods

The second hyperbolic equation-based restoration model, which was introduced in [11], has the following form:

$$\begin{cases} \alpha^2 \frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} - \frac{\gamma^2}{2} \Delta u + E \cdot \nabla u = 0 \\ u(0, x, y) = u_0(x, y), (x, y) \in \Omega \\ u(t, x, y) = 0, \quad (x, y) \in \partial\Omega \end{cases} \tag{10}$$

where $\alpha, \beta, \gamma \in (0,1]$, and the function $E : R^2 \to R^2$ is:

$$E(x, y) = \left(e^{-\eta(x^2+y^2)}, e^{-\xi(x^2+y^2)}\right) \tag{11}$$

with $\eta, \xi > 0$.

This also represents a well-posed PDE model. Since it has a unique weak solution that is propagating with finite speed [11], (10) is a non-Fourier model for the heat propagation, too. The variational solution of the model (10) is determined using the next finite-difference based iterative explicit numerical approximation scheme that is consistent to the hyperbolic PDE model [11]:

$$\begin{aligned} u^{n+1}(i, j) &= \frac{4\alpha^2}{2\alpha^2 + \beta} u^n(i, j) + \frac{\beta - 2\alpha^2}{2\alpha^2 + \beta} u^{n-1}(i, j) + \\ &\frac{\gamma^2}{2\alpha^2 + \beta}\left(u^n(i+1, j) + u^n(i-1, j) + u^n(i, j+1) + u^n(i, j-1) - 4u^n(i, j)\right) - \\ &\frac{2}{2\alpha^2 + \beta}\left(e^{-\eta(i^2+j^2)}, e^{-\xi(i^2+j^2)}\right) \cdot \left(\frac{u^n(i+1, j) + u^n(i-1, j)}{2}, \frac{u^n(i, j+1) + u^n(i, j-1)}{2}\right) \end{aligned} \tag{12}$$

The discretization algorithm (12) was applied on many noisy images. It removed successfully the additive Gaussian noise and preserve well the edges, corners and other details. Also, the obtained linear PDE-based filtering algorithm avoids unintended effects like blurring, staircasing [21] and speckle noise [8]. The method comparison described in Table 2 and Fig. 3 shows the effectiveness of our hyperbolic PDE-based filter that outperforms both classic 2D image filters and several nonlinear PDE-based smoothing models [11].

Table 2. PSNR scores of the denoising methods

The proposed hyperbolic PDE –based model	26.81 (dB)
Gaussian 2D	22.38 (dB)
Average filter	23.17 (dB)
Median filter	23.92 (dB)
Wiener 2D filter	24.73 (dB)
Perona-Malik model	25.89 (dB)
TV – ROF Denoising	25.24 (dB)

Fig. 3 Barbara image denoised by several filters

Another linear PDE-based image filtering model was derived from a stochastic differential equation (SDE) [12]. So, the following SDE-based additive Gaussian denoising model was proposed by us:

$$\begin{cases} dX(t) + F(X(t))dt = dW(t) \\ X(0,x,y) = X_0(x,y) \in R^2 \end{cases} \quad (13)$$

where the diffusion process $X(t) = \{X_1(t), X_2(t)\}$ and $W(t) = \mu\{\beta_1(t), \beta_2(t)\}, \mu \in (0,1)$ represents the 2*D* Brownian motion in the probability space $\{\Omega, F, P\}$ with natural filtration (F_t), $t \geq 0$ [12]. The restored image *u* is determined as:

$$u(t, X_0(x,y)) = E[u_0(X(t), X_0(x,y))], t \geq 0 \quad (14)$$

where *E* is the expectation operation. Then, by applying the Kolmogorov equation corresponding to the SDE-based model (13), one obtains the next linear parabolic PDE model:

$$\begin{cases} \frac{\partial u}{\partial t}(t,\xi) = \frac{\mu^2}{2}\Delta_\xi u(t,\xi) - F(\xi)\cdot\nabla_\xi u(t,\xi), t \geq 0 \\ u(0,\xi) = u_0(\xi), \xi \in R^2 \end{cases} \quad (15)$$

where $\xi = X_0(x,y) = \{(i,j)\}_{i=\overline{1,M}, j=\overline{1,N}} \in R^2$ and $\mu \in (0,1]$ [12]. The drift term F was properly modeled, such that to determine optimal filtering results, as $F(X_1(t), X_2(t)) = \left(e^{-\alpha_1(X_1(t)^2+X_2(t)^2)}, e^{-\alpha_2(X_1(t)^2+X_2(t)^2)}\right)$, where $\alpha_1, \alpha_2 \geq 0$.

The following finite difference-based iterative explicit numerical approximation algorithm was constructed for the PDE model:

$$\begin{aligned} u^{n+1}(i,j) &= (\lambda - 2\mu^2 + 1)u^n(i,j) + \frac{\mu^2}{2}\left(u^n(i+1,j) + u^n(i-1,j) + u^n(i,j+1) + u^n(i,j-1)\right) + \\ &- e^{-\alpha_1(i^2+j^2)}\frac{u^n(i+1,j) - u^n(i-1,j)}{2} - e^{-\alpha_2(i^2+j^2)}\frac{u^n(i,j+1) - u^n(i,j-1)}{2} - \lambda u^0(i,j) \end{aligned} \quad (16)$$

This approximation scheme is stable and consistent to the SDE-derived differential model (15). It was applied succesfully in our denoising and restoration experiments. The proposed linear parabolic PDE-based filtering solution outperforms the conventional filters and some diffusion-based schemes, as illustrated by the average PSNR scores in the next table.

Table 3. Average PSNR values achieved by various filters

The proposed PDE-based filter	26.94 (dB)
Gaussian filter	22.43 (dB)
Average filter	23.29 (dB)
Wiener 2D filter	24.23 (dB)
Perona-Malik 1	25.69 (dB)
Perona-Malik 2	25.83 (dB)
TV – ROF Denoising	24.96 (dB)

4. Conclusions

An overview on the linear PDE-based Gaussian noise removal techniques has been provided in this work. The image filtering methods using various linear diffusion equations have been described, their main advantages and disadvantages being discussed. They provide better restoration results than the conventional filters, but are outperformed by the nonlinear PDE-based techniques in terms of the performance measures. However, they have a much lower computational complexity than the nonlinear diffusion-based denoising methods and a shorter execution time. Also, they avoid some unintended effects that could be generated by the nonlinear PDEs, such as the blocky (staircase) effect and the multiplicative speckle noise.

Our main contributions in this image restoration field were also discussed here. The improved linear PDE-based denoising techniques proposed by us address properly the shortcomings of the existing linear PDE-based filters, such as the blurring effect, the absence of the localization property and the edge dislocation. The two hyperbolic PDE-based restoration approaches developed by us not only preserve the images edges and other essential details during the denoising process, but also sharpens them, given the second derivative of the evolving image function in these hyperbolic equations. Also, our filtering techniques outperform not only the well-known classic image filters, but also some variational and nonlinear anisotropic diffusion models, as shown by the described method comparison results. However, they are outperformed by the state of the art nonlinear PDE-based denoising approaches.

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