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FIRST ORDER STEP RESPONSE IDENTIFICATION FROM NOISY DATA

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Abstract. *In the real world, systems and signals are affected by stochastic perturbations briefly referred to as „noises". Such noises can be generated by unknown sources located either inside or outside of a system and usually corrupt in unknown way the acquired data the system can provide. Depending on the noises power in the acquired data, some characteristics of the system under study can or cannot be determined. This article introduces a method to identify optimal smooth step response, of first order, from noisy data, by means of Newton-Raphson method employed to minimize a quadratic criterion. Simulations with real world data prove the method effectiveness.*

Keywords: step response, quadratic criterion, Newton-Raphson optimization procedure

1. Introduction and problem statement

Automatic Control is a scientific and engineering field that strongly relies on Systems Theory [7], [3], [6], System Identification [8], [4] and Optimization Theory [1], [9], as main pillars. Since its inception, despite the very complex theoretical approach that the above-mentioned theories have reached, a humongous number of automatic control applications were developed. Usual practitioners are mostly interested in applying rather simple theoretical results to real world systems. Nevertheless, lately, in some applications, there is an increasing interest of filling the gap between advanced (often complex) theoretical results and numerical procedures that can be implemented on their basis. For example, in Industrial Automation field [5], very seldom controllers outside the PID class are accepted, whilst, in Aerospace Industry [2], optimal state space controllers are already implemented, although their complexity is sensibly higher.

This article tries to answer a question of interest for many Automatic Control practitioners, namely: how to extract some characteristics of a dynamic system from real world, by using acquired data the system can provide? At a first sight, the problem related to this question should not be so difficult to solve. Nevertheless, complications arise because the signals of real world are corrupted by stochastic perturbations, also referred to as *noises*. Unfortunately, in most cases, one cannot know what the noise sources are, whether they are located

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inside or outside the system under study and how the noises are mixed with the signals to acquire. Or, especially in Systems Theory, many theoretical results are concerned with dynamical entities that evolve in a *clean* environment, either noise free or with very small influence of noises. Albert Einstein expressed long time ago a sad thought: "It always distresses me to see how a beautiful theory is destroyed by an ugly reality…". Hopefully, although not perfect, there are scientific tools allowing the user to draw some line between the useful information the acquired data encode and the parasite information induced by noises in the data.

In this article, only a small part of the general problem stated above is approached. *Expressis verbis*, one seeks to build a smooth first order step response from a dataset corrupted by noises. Such a response is displayed in Figure 1 and exhibits the variation of air flow temperature acquired at the output of a climatizer. As one can easily notice, this response suggests that, maybe, the climatizer can be modelled as a first order system that responded to some step input signal. However, the theoretical (smooth) first order step response is not easy to draw empirically.

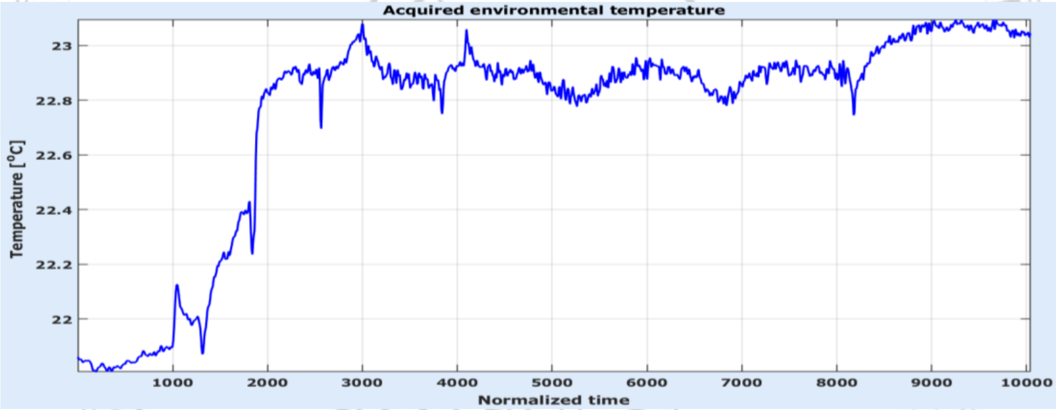

Figure 1. Noisy temperature data acquired from a climatizer output

The smooth step response could help the user to extract some practical features of the dynamical system that yielded acquiring the dataset. More specifically, from such a response, one can estimate the intrinsic delay τ, the time constant T, the DC gain K and the initial offset Δy, as shown in Figure 2. Subsequently, the 4 parameters can help the user to tune an appropriate or optimal controller for the climatizer (e.g. of PID type, by using Broida method [10]).

To determine the best step response for a given dataset, an optimization criterion is needed. In this case, the usual quadratic criterion can be employed. Its minimization can be performed by means of various exact optimization techniques [1].

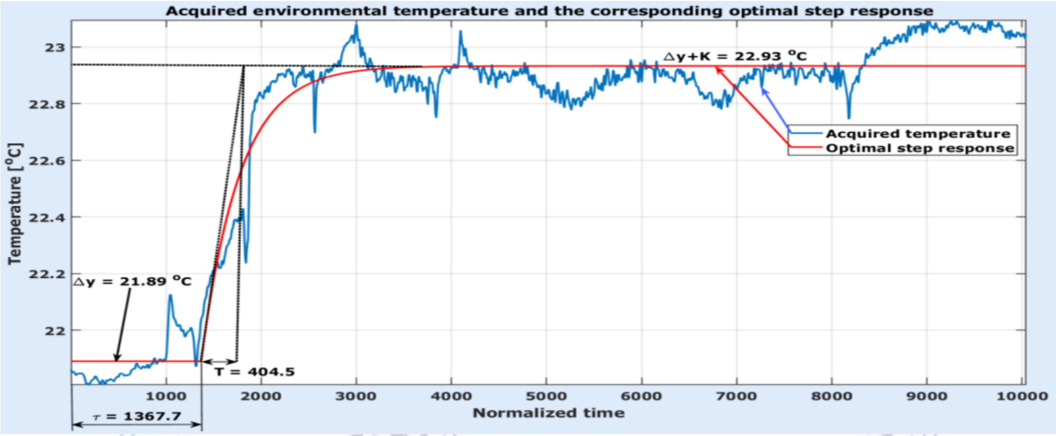

Figure 2. Theoretical first order step response and its main parameters

Such a technique is the *Newton-Raphson*'s (N-R) one, due to the possibility to compute not only the gradient (or the first derivative) of quadratic criterion, but also its Hessian matrix (or the second derivative).

The article is structured as follows. The next section is devoted to mathematical formulation of optimization problems to solve. In section 3, one shows how the optimal solution can theoretically be obtained, by means of N-R numerical procedure. Some simulation results on real world data are presented and analyzed in Section 4. The article completes with concluding remarks and a references list.

2. Optimization problems to solve

One starts with the mathematical expression of smooth (noise free) first order step response, in continuous time:

$$y(t) = \begin{cases} \Delta y & , \ t \in [0, \tau) \\ K\left(1 - e^{-\frac{t-\tau}{T}}\right) + \Delta y & , \ t \geq \tau \end{cases} , \tag{1}$$

where the shape is uniquely determined by 4 parameters: $\tau \geq 0$ (the intrinsic delay), $T > 0$ (the time constant), $K \in \mathbb{R}^*$ (the DC gain) and $\Delta y \in \mathbb{R}$ (the initial offset). This shape should fit to the best to the available dataset $\mathcal{D}_N = \{y_n\}_{n \in \overline{0,N-1}}$, with $N \in \mathbb{N}^*$. Obviously, the data were acquired with some sampling period, $T_s > 0$ (unit, if unknown). Consequently, before anything else, the variable part of step response (1) has to be discretized with the same period:

$$y[n] = y(nT_s) = K\left(1 - e^{-\frac{nT_s-\tau}{T}}\right) + \Delta y, \quad \forall n \geq n_\tau = \left\lceil \frac{\tau}{T_s} \right\rceil. \tag{2}$$

Then, the following quadratic criterion can be employed to determine the best fitting step response to the dataset $\mathcal{D}_N$:

$$\mathcal{V}_N(\tau,T,K,\Delta y)=\sum_{n=0}^{n_\tau-1}\left(y_n-\Delta y\right)^2+\sum_{n=n_\tau}^{N-1}\left[y_n-K\left(1-\mathrm{e}^{-\frac{nT_s-\tau}{T}}\right)-\Delta y\right]^2. \quad (3)$$

It follows that the optimization problem can be formulated as follows:

$$\left(\hat{\tau},\hat{T},\hat{K},\Delta y\right)=\underset{\tau\geq 0,T>0,K\in\mathbb{R}^*,\Delta y\in\mathbb{R}}{\arg\min}\ \mathcal{V}_N(\tau,T,K,\Delta y). \quad (4)$$

In general, the solution of problem (4) can be found by solving the null gradient equation:

$$\nabla\mathcal{V}_N(\tau,T,K,\Delta y)=0. \quad (5)$$

However, the minimum of $\mathcal{V}_N$ is obtained when both sums of quadratic errors are minimum. This means the first sum is minimum when:

$$\frac{\partial}{\partial(\Delta y)}\left[\sum_{n=0}^{n_\tau-1}\left(y_n-\Delta y\right)^2\right]=0 \quad\Leftrightarrow\quad -2\sum_{n=0}^{n_\tau-1}\left(y_n-\Delta y\right)=0. \quad (6)$$

Thus, the optimum initial offset is:

$$\Delta y=\frac{1}{n_{\hat{\tau}}}\sum_{n=0}^{n_\tau-1}y_n, \quad (7)$$

once the optimum intrinsic delay $\hat{\tau}$ has been found. This corresponds well to the image in Figure 2.

The other optimum parameters are solutions of the compatible system below, obtained from (5):

$$\begin{cases}\displaystyle\sum_{n=n_\tau}^{N-1}\mathrm{e}^{-\frac{nT_s}{T}}\left[y_n-K\left(1-\mathrm{e}^{\frac{\tau}{T}}\mathrm{e}^{-\frac{nT_s}{T}}\right)-\Delta y\right]=0\\ \displaystyle\sum_{n=n_\tau}^{N-1}\left(1-\mathrm{e}^{\frac{\tau}{T}}\mathrm{e}^{-\frac{nT_s}{T}}\right)\left[y_n-K\left(1-\mathrm{e}^{\frac{\tau}{T}}\mathrm{e}^{-\frac{nT_s}{T}}\right)-\Delta y\right]=0.\\ \displaystyle\sum_{n=n_\tau}^{N-1}n\mathrm{e}^{-\frac{nT_s}{T}}\left[y_n-K\left(1-\mathrm{e}^{\frac{\tau}{T}}\mathrm{e}^{-\frac{nT_s}{T}}\right)-\Delta y\right]=0\end{cases} \quad (8)$$

Since the system (8) is nonlinear, the optimal solution is not easy to compute. Therefore, it is suitable to simplify the approach and to search for a sub-optimal solution, with acceptable accuracy.

This allows splitting the optimization problem in two sub-problems, by grouping the unknown parameters in two couples: $\{K,\Delta y\}$ on one side and $\{\tau,T\}$ on the other side. Two new quadratic criteria can now be defined. The equations of system (8) suggest that, to simplify the solving procedure of optimization problems, the following notations should be employed:

$$\xi = e^{-\frac{T_s}{T}} \in (0,1) \text{ and } \theta = e^{\frac{\tau}{T}} \geq 1. \tag{9}$$

Then, the new optimization criteria are:

$$\mathcal{V}_N^1(K) = \sum_{n=n_{\hat{\tau}}}^{N-1} \left[\left(y_n - \Delta y \right) - K\left(1 - \hat{\theta}\hat{\xi}^n\right) \right]^2 ; \tag{10}$$

$$\mathcal{V}_N^2(\theta,\xi) = \sum_{n=n_\tau}^{N-1} \left[\left(y_n - \Delta y - \hat{K} \right) + \hat{K}\theta\xi^n \right]^2 . \tag{11}$$

In definition (10), one considers that the couple $\left\{\hat{\tau},\hat{T}\right\}$ is already estimated, which allows computing $\hat{\theta}$ and $\hat{\xi}$ according to notations (9). Also, the discrete time delay is $n_{\hat{\tau}} = \left\lceil \hat{\tau} / T_s \right\rceil$. Similarly, in definition (11), the DC gain $\hat{K}$ is known. In both criteria, the offset Δy is computed by means of equation (7). In criterion (11), the delay τ will be estimated recursively, so that Δy can always be computed from the delay value available at previous iteration.

The criterion (10) can equivalently be expressed by performing the following index changing: $n \leftarrow n - n_{\hat{\tau}}$. Thus,

$$\mathcal{V}_N^1(K) = \sum_{n=0}^{N-n_{\hat{\tau}}-1} \left[\left(y_{n+n_{\hat{\tau}}} - \Delta y \right) - K\left(1 - \hat{\theta}\hat{\xi}^{n+n_{\hat{\tau}}}\right) \right]^2 . \tag{12}$$

Moreover, equation (12) reveals that, instead of using the truncated data $\left\{y_{n+n_{\hat{\tau}}}\right\}_{n\in\overline{0,N-n_{\hat{\tau}}-1}}$, one can work with the modified data set:

$$y_{\hat{\tau},n} = y_{n+n_{\hat{\tau}}} - \Delta y, \quad \forall n \in \overline{0, N_{\hat{\tau}} - 1}, \tag{13}$$

where $N_{\hat{\tau}} = N - n_{\hat{\tau}}$. Then, the criterion (10) becomes:

$$\mathcal{V}_N^1(K) = \sum_{n=0}^{N_{\hat{\tau}}-1} \left[y_{\hat{\tau},n} - K\left(1 - \hat{\theta}\hat{\xi}^{n+n_{\hat{\tau}}}\right) \right]^2 , \tag{14}$$

which simplifies its definition. The new definition, (14), involves that two preliminary operations have to be applied on the genuine dataset: (a) remove the subset located into the delay zone delimited by $n_{\hat{\tau}}$; (b) shift the remaining data by the offset Δy, as illustrated in Figure 3.

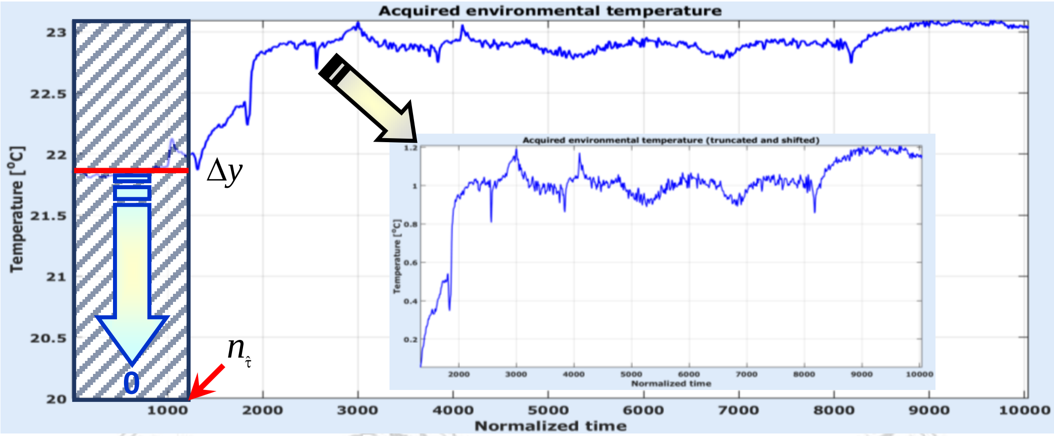

Figure 3. Preliminary operations to apply on dataset for the quadratic criterion (10)

In case of criterion (11), the approach is similar. By performing the index changing $n \leftarrow n - n_\tau$, a new expression is obtained:

$$\mathcal{V}_N^2(\theta,\xi) = \sum_{n=0}^{N-n_\tau-1}\left[\left(y_{n+n_\tau} - \Delta y - \hat{K}\right) + \hat{K}\theta\xi^{n+n_\tau}\right]^2 . \tag{15}$$

According to (15), the modified data set is:

$$\hat{y}_{\tau,K,n} = y_{n+n_\tau} - \Delta y - \hat{K}\,, \quad \forall n \in \overline{0, N_\tau - 1}, \tag{16}$$

where $N_\tau = N - n_\tau$ is the variable number of data to work with. Practically, the operations to apply on the genuine dataset are almost the same as pointed in Figure 3. One difference is that, here, the data are shifted by the sum $\Delta y + \hat{K}$ (instead of Δy), which enforces the steady-state part to vary around the time axis (provided the offset and the DC gain were correctly estimated). Another difference is given by the delay τ, which, here, is unknown.

From equation (15), one obtains:

$$\begin{aligned}\mathcal{V}_N^2(\theta,\xi) &= \sum_{n=0}^{N_\tau-1}\left(\hat{y}_{\tau,K,n} + \hat{K}\theta\xi^{n+n_\tau}\right)^2 = \\ &= \left(\hat{y}_{\tau,K,0} + \hat{K}\theta\xi^{n_\tau}\right)^2 + \sum_{n=1}^{N_\tau-1}\left(\hat{y}_{\tau,K,n} + \hat{K}\theta\xi^{n+n_\tau}\right)^2 .\end{aligned} \tag{17}$$

The final expression in (17) allows making the difference between two cases: $n_\tau = 0$ and $n_\tau > 0$. If no intrinsic delay exists ($\tau = 0$), then the first term of final expression in (17) is independent on ξ (as $\xi^0 = 1$); also, the criterion $\mathcal{V}_N^2$ does not depend on θ variable (as $\theta = \mathrm{e}^{\frac{0}{T}} = 1$). Otherwise ($\tau > 0$), the full sum in (17) can be employed (as $0 < \xi^{n_\tau} < 1$ and $\theta = \mathrm{e}^{\frac{\tau}{T}} > 1$).

The optimization problems to solve are then:

$$\hat{K} = \arg\min_{K \in \mathbb{R}^*} \mathcal{V}_N^1(K)\,; \tag{18}$$

$$\left(\hat{\theta}, \hat{\xi}\right) = \arg\min_{\theta > 1, \xi \in (0,1)} \mathcal{V}_N^2(\theta, \xi)\,. \tag{19}$$

3. Solving the optimization problems

Like in case of overall criterion (3), to solve the optimization problems (18) and (19), the gradients of criteria (14) and (17) need to be computed in the first place. Then, the solutions result by solving the equations obtained when enforcing the gradients to null values.

After some simple algebraic manipulations, the equations to solve are:

$$\sum_{n=0}^{N_{\hat\tau}-1} \left(1 - \hat{\theta}\hat{\xi}^{n+n_{\hat\tau}}\right)\left[y_{\hat\tau,n} - K\left(1 - \hat{\theta}\hat{\xi}^{n+n_{\hat\tau}}\right)\right] = 0\,. \tag{20}$$

$$\sum_{n=0}^{N_\tau - 1} \xi^{n+n_\tau}\left(\hat{y}_{\tau,K,n} + \hat{K}\theta\xi^{n+n_\tau}\right) = 0 \;\; (\tau > 0 \Rightarrow \theta > 1)\,. \tag{21}$$

$$\begin{cases} \displaystyle\sum_{n=1}^{N-1} n\xi^{n-1}\left(y_n + \hat{K}\xi^n\right) = 0 & ,\; n_\tau = 0 \;\; (\tau = 0 \Rightarrow \theta = 1) \\ \displaystyle\sum_{n=0}^{N_\tau - 1} (n + n_\tau)\xi^{n+n_\tau-1}\left(\hat{y}_{\tau,K,n} + \hat{K}\theta\xi^{n+n_\tau}\right) = 0 & ,\; n_\tau > 0 \;\; (\tau > 0 \Rightarrow \theta > 1) \end{cases}. \tag{22}$$

Note that, in no delay case, since the criterion (17) only depends on ξ variable, the system (21)-(22) is replaced by the first equation of (22).

Focus first on equation (20), which directly gives:

$$\hat{K} = \frac{\displaystyle\sum_{n=0}^{N_{\hat\tau}-1} \left(1 - \hat{\theta}\hat{\xi}^{n+n_{\hat\tau}}\right) y_{\hat\tau,n}}{\displaystyle\sum_{n=0}^{N_{\hat\tau}-1} \left(1 - \hat{\theta}\hat{\xi}^{n+n_{\hat\tau}}\right)^2}\,. \tag{23}$$

To yield efficient implementation of solution (23), the sums of geometric series in denominator can be computed in advance. Thus:

$$\sum_{n=0}^{N_{\hat\tau}-1} \left(1 - \hat{\theta}\hat{\xi}^{n+n_{\hat\tau}}\right)^2 = N_{\hat\tau} - 2\hat{\theta}\hat{\xi}^{n_{\hat\tau}} \sum_{n=0}^{N_{\hat\tau}-1} \hat{\xi}^n + \hat{\theta}^2\hat{\xi}^{2n_{\hat\tau}} \sum_{n=0}^{N_{\hat\tau}-1} \hat{\xi}^{2n} =$$

$$= N_{\hat\tau} - 2\hat{\theta}\hat{\xi}^{n_{\hat\tau}} \frac{\hat{\xi}^{N_{\hat\tau}} - 1}{\hat{\xi} - 1} + \hat{\theta}^2\hat{\xi}^{2n_{\hat\tau}} \frac{\hat{\xi}^{2N_{\hat\tau}} - 1}{\hat{\xi}^2 - 1} =$$

$$= N_{\hat{\tau}} + \hat{\theta}\hat{\xi}^{n_{\hat{\tau}}}\,\frac{\hat{\xi}^{N_{\hat{\tau}}}-1}{\hat{\xi}-1}\left(\hat{\theta}\hat{\xi}^{n_{\hat{\tau}}}\,\frac{\hat{\xi}^{N_{\hat{\tau}}}+1}{\hat{\xi}+1}-2\right). \tag{24}$$

Approach now the system (21)-(22).

a. If $\tau = 0$ ($\theta = 1$), only the first equation of (22) is available:

$$\sum_{n=1}^{N-1} n\xi^{n-1} y_n = -\hat{K}\sum_{n=1}^{N-1} n\xi^{2n-1}. \tag{25}$$

The sum in the right side of equation (25) can be computed by using geometric series. Hence:

$$\begin{aligned}\sum_{n=1}^{N-1} n\xi^{2n-1} &= \xi\sum_{n=1}^{N-1} n\left(\xi^2\right)^{n-1} = \xi\frac{d}{d\left(\xi^2\right)}\left(\sum_{n=0}^{N-1}\left(\xi^2\right)^n\right) = \xi\frac{d}{d\left(\xi^2\right)}\left(\frac{\left(\xi^2\right)^N-1}{\xi^2-1}\right) = \\ &= \xi\frac{N\left(\xi^2\right)^{N-1}\left(\xi^2-1\right)-\left(\xi^2\right)^N+1}{\left(\xi^2-1\right)^2} = \xi\frac{(N-1)\xi^{2N}-N\xi^{2N-2}+1}{\left(\xi^2-1\right)^2}.\end{aligned} \tag{26}$$

By inserting (26) in (25), the equation to solve becomes:

$$\left(\xi^2-1\right)^2\sum_{n=1}^{N-1} n\xi^{n-1} y_n = -\hat{K}\xi\left[(N-1)\xi^{2N}-N\xi^{2N-2}+1\right]. \tag{27}$$

b. If $\tau > 0$, the product $\hat{K}\theta$ can be expressed from (21) and second equation of (22):

$$\hat{K}\theta = -\frac{\displaystyle\sum_{n=0}^{N_\tau-1}\xi^{n+n_\tau}\hat{y}_{\tau,K,n}}{\displaystyle\sum_{n=0}^{N_\tau-1}\xi^{2(n+n_\tau)}} = -\frac{\displaystyle\sum_{n=0}^{N_\tau-1}\left(n+n_\tau\right)\xi^{n+n_\tau-1}\hat{y}_{\tau,K,n}}{\displaystyle\sum_{n=0}^{N_\tau-1}\left(n+n_\tau\right)\xi^{2(n+n_\tau)-1}} \quad\Leftrightarrow$$

$$\Leftrightarrow\quad \frac{\displaystyle\sum_{n=0}^{N_\tau-1}\xi^{n}\hat{y}_{\tau,K,n}}{\displaystyle\sum_{n=0}^{N_\tau-1}\xi^{2n}} = \frac{\displaystyle\sum_{n=0}^{N_\tau-1}\left(n+n_\tau\right)\xi^{n}\hat{y}_{\tau,K,n}}{\displaystyle\sum_{n=0}^{N_\tau-1}\left(n+n_\tau\right)\xi^{2n}} \quad\Leftrightarrow$$

$$\Leftrightarrow\quad \left(\xi^2\sum_{n=1}^{N_\tau-1} n\xi^{2n-2} + n_\tau\sum_{n=0}^{N_\tau-1}\xi^{2n}\right)\sum_{n=0}^{N_\tau-1}\xi^n\hat{y}_{\tau,K,n} = \left(\sum_{n=0}^{N_\tau-1}\xi^{2n}\right)\sum_{n=0}^{N_\tau-1}\left(n+n_\tau\right)\xi^n\hat{y}_{\tau,K,n} \quad\Leftrightarrow$$

$$\Leftrightarrow\quad \xi\left(\sum_{n=1}^{N_\tau-1} n\xi^{2n-2}\right)\sum_{n=0}^{N_\tau-1}\xi^n\hat{y}_{\tau,K,n} = \left(\sum_{n=0}^{N_\tau-1}\xi^{2n}\right)\sum_{n=1}^{N_\tau-1} n\xi^{n-1}\hat{y}_{\tau,K,n}. \tag{28}$$

Equation (28) can be transformed by means of geometric series (see property (26) as well):

$$\xi\left[\left(N_\tau-1\right)\xi^{2N_\tau}-N\xi^{2N_\tau-2}+1\right]\sum_{n=0}^{N_\tau-1}\xi^n\hat{y}_{\tau,K,n} =$$

$$= \left(\xi^2 - 1\right)\left(\xi^{2N_\tau} - 1\right)\sum_{n=1}^{N_\tau - 1} n\xi^{n-1}\hat{y}_{\tau,K,n}. \quad (29)$$

None of equations (27) or (29) can be solved in closed form, as the unknown variable ξ cannot be extracted from the sums with terms including data samples. Therefore, the N-R procedure can be employed to approximate both solutions.

Recall that the N-R procedure operates with gradients and Hessian matrices of a cost function. In case of equations (27) or (29), the corresponding cost function only depends on scalar ξ. The first two derivatives of cost function are sufficient to run the procedure, without actually knowing the cost function itself. Fortunately, each equation leads to the definition of cost function first derivative. More specifically:

a. if $\tau = 0$, then:

$$f_0'(\xi) = \left(\xi^2 - 1\right)^2 \sum_{n=1}^{N-1} n\xi^{n-1} y_n + \hat{K}\xi\left[\left(N-1\right)\xi^{2N} - N\xi^{2N-2} + 1\right]; \quad (30)$$

b. otherwise:

$$\begin{aligned} f_\tau'(\xi) = {} & \left(\xi^2 - 1\right)\left(\xi^{2N_\tau} - 1\right)\sum_{n=1}^{N_\tau - 1} n\xi^{n-1}\hat{y}_{\tau,K,n} - \\ & - \xi\left[\left(N_\tau - 1\right)\xi^{2N_\tau} - N_\tau\xi^{2N_\tau - 2} + 1\right]\sum_{n=0}^{N_\tau - 1}\xi^n \hat{y}_{\tau,K,n}, \end{aligned} \quad (31)$$

for any $\tau > 0$.

From (30) and (31), the second derivatives can be computed straightforwardly:

a. if $\tau = 0$, then:

$$\begin{aligned} f_0''(\xi) = {} & 2\xi\left(\xi^2 - 1\right)\sum_{n=1}^{N-1} n\xi^{n-1} y_n + \left(\xi^2 - 1\right)^2 \sum_{n=2}^{N-1} n(n-1)\xi^{n-2} y_n + \\ & + \hat{K}\left[\left(N-1\right)(2N+1)\xi^{2N} - N(2N-1)\xi^{2N-2} + 1\right]; \end{aligned} \quad (32)$$

b. otherwise:

$$\begin{aligned} f_\tau''(\xi) = {} & \xi\left[\left(N_\tau + 3\right)\xi^{2N_\tau} - N_\tau\xi^{2N_\tau - 2} - 3\right]\sum_{n=1}^{N_\tau - 1} n\xi^{n-1}\hat{y}_{\tau,K,n} + \\ & + \left(\xi^2 - 1\right)\left(\xi^{2N_\tau} - 1\right)\sum_{n=2}^{N_\tau - 1} n(n-1)\xi^{n-2}\hat{y}_{\tau,K,n} - \\ & - \xi^{2N_\tau - 2}\left[\left(N_\tau - 1\right)\left(2N_\tau + 1\right)\xi^2 - N_\tau\left(2N_\tau - 1\right)\right]\sum_{n=0}^{N_\tau - 1}\xi^n \hat{y}_{\tau,K,n}. \end{aligned} \quad (33)$$

Using the N-R procedure is quite easy in case of null intrinsic delay. However, if $\tau > 0$, the main problem is the dependency of both derivatives on that delay. Therefore, to allow implementation of N-R procedure, the following strategy could be adopted.

A. If the initial time delay is null, then only equations (30), (32) and (23)-(24) can be employed. In this case, n_τ always is null and only parameters $\{K,T\}$ can be estimated.

B. If the initial time delay is non null:

1. Use the (initial) value of $\hat{\tau} > 0$ to find the (first) value $\hat{\xi}$, by means of N-R procedure (equations (31) and (33)).
2. With $\hat{\xi}$, the parameter θ can be estimated by using equation (21):

$$\hat{\theta} = -\frac{1}{\hat{K}\hat{\xi}^{n_\tau}} \frac{\sum_{n=0}^{N_\tau-1} \hat{\xi}^n \hat{y}_{\tau,K,n}}{\sum_{n=0}^{N_\tau-1} \hat{\xi}^{2n}} = \frac{\left(1-\hat{\xi}^2\right)\sum_{n=0}^{N_\tau-1} \hat{\xi}^n \hat{y}_{\tau,K,n}}{\hat{K}\hat{\xi}^{n_\tau}\left(\hat{\xi}^{2N_\tau}-1\right)} . \tag{34}$$

3. From $\hat{\xi}$ and $\hat{\theta}$, the delay can be estimated with equations (9):

$$\hat{T} = -\frac{T_s}{\log \hat{\xi}} \quad \Rightarrow \quad \hat{\tau} = \hat{T} \log \hat{\theta} . \tag{35}$$

4. Estimate the initial offset Δy by means of equation (7).
5. Use equation (23) to estimate the DC gain, $\hat{K}$.
6. Repeat steps 1-5 with until the accuracy condition is met.

The model performance can be assessed by means of criterion (3) or, even better, by means of *signal-to-noise ratio* (SNR), computed with the help of criterion (3):

$$\mathrm{SNR}\left(\hat{\tau},\hat{T},\hat{K},\Delta y\right) = \left. \frac{\frac{1}{N}\sum_{n=0}^{N-1}\left(y_n - \overline{y}\right)^2}{\frac{1}{N}\mathcal{V}_N\left(\hat{\tau},\hat{T},\hat{K},\Delta y\right)} \right|_{\mathrm{dB}} . \tag{36}$$

In definition (36), $\overline{y}$ is the noisy data average. Also, the SNR is expressed in decibels (dB). (Recall that $a\big|_{\mathrm{dB}} = 20\log(a)$.) Thus, the SNR is the ratio between the standard deviation of data and standard deviation of modeling error. The higher the SNR, the better the model.

The algorithm to solve the optimization problems is listed next.

Algorithm to find optimal step response, based on Newton-Raphson procedure

➢ **Inputs:**

- A noisy dataset: $\mathcal{D}_N = \{y_n\}_{n\in\overline{0,N-1}}$, with $N \in \mathbb{N}^*$.
- Sampling period: $T_s > 0$ (unit, if unknown).
- Initial values of parameters: K_0, T_0, τ_0.
- Accuracy threshold: ε (equal to 10^{-3}, by default).

❖ **Initialization**

- Compute the discrete-time delay $n_0 = \lceil \tau_0 / T_s \rceil$.
- Compute $\xi_0 = \exp(-T_s / T_0)$.
- Set the initial variable step in N-R procedure: $\alpha_0 = 1$.
- If $\tau_0 = 0$ ($\Rightarrow n_0 = 0$):
  - set $\hat{\tau} = 0$ and $\Delta y = 0$, as they cannot be estimated, in this case;
  - compute $f_0'(\xi_0)$ with (30) and $f_0''(\xi_0)$ with (32), for $\hat{K} = K_0$;
  - estimate the first correction: $\Delta_0 = -\dfrac{f_0'(\xi_0)}{f_0''(\xi_0)}$.
- Otherwise ($\tau_0 > 0 \Rightarrow n_0 > 0$):
  - compute $N_0 = N - n_0$ and $\Delta y_0 = \dfrac{1}{n_0}\sum_{n=0}^{n_0-1} y_n$;
  - extract the data beyond n_0: $y_{0,n} = y_{n+n_0} - \Delta y_0 - K_0$, $\forall n \in \overline{0, N_0 - 1}$;
  - compute $f_\tau'(\xi_0)$ with (31) and $f_\tau''(\xi_0)$ with (33) for $N_\tau = N_0$ and $\hat{y}_{\tau,K,n} = y_{0,n}$, $\forall n \in \overline{0, N_0 - 1}$.
  - estimate the first correction: $\Delta_0 = -\dfrac{f_\tau'(\xi_0)}{f_\tau''(\xi_0)}$.
- Set the initial number of iterations: $k = 0$

◎ **If $\tau_0 = 0$, do:**

1. Approach the optimum: $\xi_{k+1} = \xi_k + \alpha_k \Delta_k$.
2. Update the DC gain K_{k+1} with (23)-(24), for $\hat{\xi} = \xi_{k+1}$. (Note that, in this case, $\theta = 1$ and $N_{\hat{\tau}} = N$.)

3. Compute $f_0'(\xi_{k+1})$ with (30) and $f_0''(\xi_{k+1})$ with (32). (Note that, in this case, $\hat{K} = K_{k+1}$.)
4. Update the correction: $\Delta_{k+1} = -\dfrac{f_0'(\xi_{k+1})}{f_0''(\xi_{k+1})}$.
5. Update the variable step: $\alpha_{k+1} = \alpha_k + \dfrac{\Delta_{k+1}}{\Delta_k}$.
6. Increment the number of iterations: $k \leftarrow k+1$.

◎ **until** $|\Delta_k| < \varepsilon$.

◎ **If** $\tau_0 > 0$**, do:**

1. Approach the optimum: $\xi_{k+1} = \xi_k + \alpha_k \Delta_k$.
2. Compute θ_{k+1} with (34), for $\hat{\xi} = \xi_{k+1}$. (Note that, in this case, $\hat{K} = K_k$, $n_\tau = n_k$ and $\hat{y}_{\tau,K,n} = y_{k,n}$, $\forall n \in \overline{0, N_0 - 1}$.)
3. Update $T_{k+1} = -T_s / \log \xi_{k+1}$ and $\tau_{k+1} = T_{k+1} \log \theta_{k+1}$.
4. Update: $n_{k+1} = \lceil \tau_{k+1} / T_s \rceil$ and $N_{k+1} = N - n_{k+1}$.
5. Update the initial offset: $\Delta y_{k+1} = \dfrac{1}{n_{k+1}} \sum_{n=0}^{n_{k+1}-1} y_n$.
6. Extract the data beyond n_{k+1}:
$$y_{k+1,n} = y_{n+n_{k+1}} - \Delta y_{k+1} - K_k, \ \forall n \in \overline{0, N_{k+1} - 1}.$$
7. Update the DC gain K_{k+1} with (23)-(24), for $\hat{\xi} = \xi_{k+1}$, $\theta = \theta_{k+1}$ and $N_{\hat{\tau}} = N_{k+1}$.
8. Compute $f_\tau'(\xi_{k+1})$ with (31) and $f_\tau''(\xi_{k+1})$ with (33). (Note that, in this case, $N_\tau = N_{k+1}$ and $\hat{y}_{\tau,K,n} = y_{k+1,n}$, $\forall n \in \overline{0, N_0 - 1}$.)
9. Update the correction: $\Delta_{k+1} = -\dfrac{f_\tau'(\xi_{k+1})}{f_\tau''(\xi_{k+1})}$.
10. Update the variable step: $\alpha_{k+1} = \alpha_k + \dfrac{\Delta_{k+1}}{\Delta_k}$.
11. Increment the number of iterations: $k \leftarrow k+1$.

◎ **until** $|\Delta_k| < \varepsilon$.

❖ **Final computations**

- Set: $\hat{K} = K_k$, $\hat{T} = T_k$.
- If $\tau_0 > 0$, set: $\hat{\tau} = \tau_k$, $\Delta y = \Delta y_k$.
- Compute the optimal step response $\hat{y}$ with (1) and (2), by using the sampling period T_s beyond $\lfloor \hat{\tau} / T_s \rfloor$.
- Estimate the step response performance by computing $\mathrm{SNR}\left(\hat{\tau}, \hat{T}, \hat{K}, \Delta y\right)$ with (36).

≺ **Outputs:**

- Optimal values of parameters: $\hat{K}$, $\hat{T}$, $\hat{\tau}$, Δy.
- Optimal step response, sampled with T_s: $\hat{\mathcal{D}}_N = \{\hat{y}_n\}_{n \in \overline{0,N-1}}$, with same length as the initial dataset (N).
- Step response performance: $\mathrm{SNR}\left(\hat{\tau}, \hat{T}, \hat{K}, \Delta y\right)$ (in dB).
- Number of iterations to meet the accuracy condition: k.

4. Simulation results and discussion

As already mentioned in Introduction, the dataset to test was acquired at the output of a climatizer mounted in a large room, with many sources of noises (such as doors and windows that are opened and closed very frequently). The temperature of airflow was measured and recorded with the sampling period $T_s = 1\ \mathrm{min}$. The number of samples is $N = 10041$ (quite big). Figure 1 displays the variation of temperature. One can notice the noises are quite powerful, so that it is difficult to manually draw a fitted step response.

The algorithm above has been implemented into MATLAB™ programming environment. The procedure was initiated to run with the initialization below:

$$K_0 = 1\,^\circ\mathrm{C},\ T_0 = 430\ \mathrm{min},\ \tau_0 = 1000\ \mathrm{min},\ \varepsilon = 10^{-3}, \tag{37}$$

which has been empirically obtained, like in Figure 3, after visually inspecting the variation in Figure 1. The optimization results are illustrated in Figure 2, where the optimal parameters are:

$$\hat{K} \cong 1.043\,^\circ\mathrm{C},\ \hat{T} \cong 404.5\ \mathrm{min},\ \hat{\tau} \cong 1367.7\ \mathrm{min},\ \Delta y \cong 21.89\,^\circ\mathrm{C}. \tag{38}$$

Moreover, the steady-state temperature can also be estimated:

$$\hat{K} + \Delta y \cong 22.93\,^\circ\mathrm{C} \tag{39}$$

Apparently, the climatizer tries to keep the room temperature as close as possible to the value of $23\,^{\circ}\mathrm{C}$.

The performance of N-R procedure is revealed not only by the small number of iterations until the optimal values were found, $k = 12$, but also by the search duration, $\Delta T \cong 0.32\,\mathrm{s}$ (as obtained on a personal computer from i7 family, of 10-th generation, with 8 parallel processors).

The performance of optimal step response is given by:

$$\mathrm{SNR}\left(\hat{\tau}, \hat{T}, \hat{K}, \Delta y\right) \cong 12.987\,\mathrm{dB}. \tag{40}$$

The main limitation of the algorithm above is its sensitivity to initialization. The simulations revealed that the procedure is the most sensitive to small variations of initial DC gain values. For example, if the initial DC gain decreases to $K_0 = 0.9\,^{\circ}\mathrm{C}$ (with $0.1\,^{\circ}\mathrm{C}$ only), while keeping the other initial values, the sub-optimal result of Figure 4 is obtained. Comparing to the optimal result in Figure 2, this step response has smaller performance, which can be observed easily.

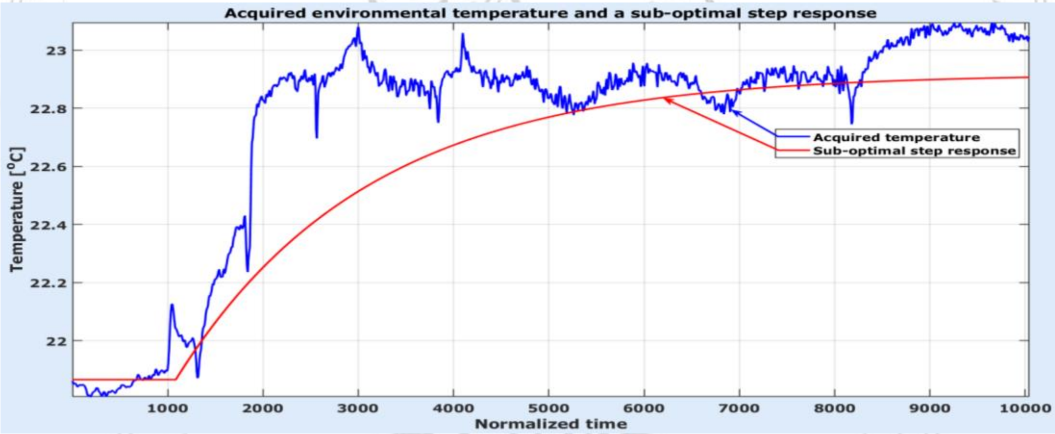

Figure 4. Sub-optimal step response, obtained after decreasing the initial DC gain by 0.1°C.

Moreover, the SNR decrease confirms the visual observation:

$$\mathrm{SNR}\left(\hat{\tau}, \hat{T}, \hat{K}, \Delta y\right) \cong 7.11\mathrm{dB}. \tag{41}$$

The performance deteriorated with more than 45%, although the initial value of DC gain decreased by 10% only. So, it is important to correctly set the initial values of parameters. To this respect, as a future development, one can use a metaheuristic approach [9] to solve the following optimization problem: what initialization leads to the highest SNR?

When looking back at the variation of temperature, even more closely, one realizes that, beside the first jump of about 1°C, in the steady-state zone, the internal regulator of climatizer seems to be a rudimentary one, working in steps. Thus, the temperature variation includes several small successive step responses, as reaction to that regulator. Then, the optimization algorithm can be employed to build the noise free response of climatizer, under the control of regulator. Evidently, this attempt involves building a set of local initializations, one for each zone where a small step response seems to exist. Although this activity could be tedious, if the initializations are realistic, the reward is guaranteed.

After running the algorithm for each possible local step response, the result is displayed in Figure 5.

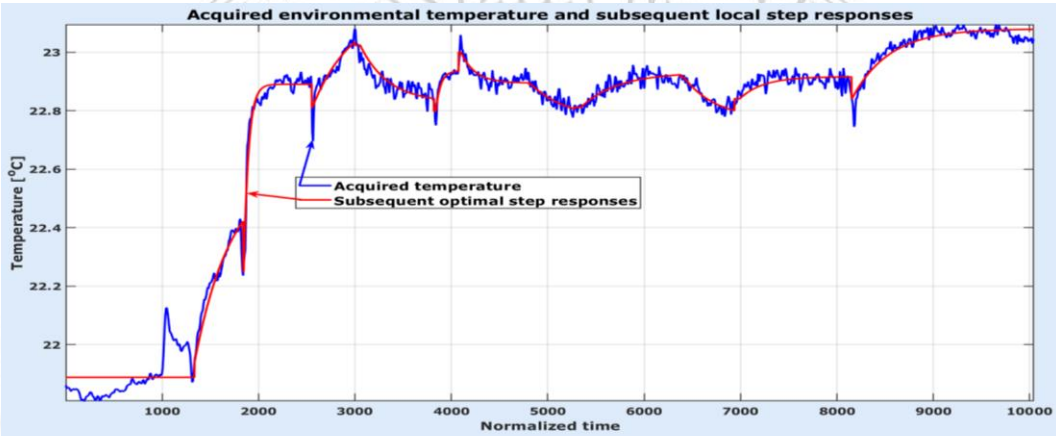

Figure 5. Optimal step responses, obtained after running the N-R procedure on each local variation zone of temperature.

The smooth variation shows that the climatizer has different parameters of step response, depending on the height of reference step (or the setpoint) to follow. This behavior was expected, as the static characteristic of such a climatizer is nonlinear and exhibits hysteresis phenomenon. Hence, changing from a nominal point to another, on the static characteristic, automatically involves changing the parameters of step response (especially of time constant). For the multiple step responses in Figure 5, denoted by $\tilde{y}$, the performance is:

$$\mathrm{SNR}(\tilde{y}) \cong 20.54\,\mathrm{dB}. \tag{42}$$

There is no wonder the SNR has sensibly increased, comparing to single step response case, as the multiple step responses curve better fits the dataset.

Concluding remarks

This article approached the problem of optimal first order step responses estimation by using noisy datasets. The proposed method to solve the problem relies on quadratic criteria and Newton-Raphson procedure. The simulation results have proven both the method effectiveness and its main limitation, namely the high sensitivity to initialization of some parameters (especially of DC gain). This drawback suggests using a metaheuristic to find the initialization that maximizes the SNR. Another possible future development is to design a method allowing the user to find the optimal second order step response from noisy data. It is well known that smooth first and second order step responses are extremely useful in Systems Theory and Industrial Automation.

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